# Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator

Kappeler, T ; Mityagin, B

Abstract: Consider the Schrödinger equation -y ” +Vy= y for a complex-valued potential V of period 1 in the weighted Sobolev space H w of 2-periodic functions f: → , H w H w :=f(x)= k=-∞ ∞ f ^(k)eikx|fw < ,wherefw := 2kw(k)2|f^(k)|21/2andw = (w(k))kdenotesasymmetric, submultiplicativeweightsequence.Denotebyn = n(V)(n0)theperiodiceigenvaluesof−d2dx2+Vwhenconsideredontheinterval[0, 2], listedinsuchawaythat2n, 2n−1 = n22+0(1), anddenotebyn = n(V)(n1)theDirichleteigenvaluesof−d2dx2+Vconsideredon[0, 1], listedinsuchawaythat n22+0(1).Theorem.Thereexist(absolute)constantsK1, K2 > 0, sothatforany1−periodicpotentialVinHw, nNw(2n)2|2n 2n − 1|2K1(1 + Vw)K2andnNw(2n)2|n − 2n|2K1(1 + Vw)K2, whereN := K1(1 + Vw)2.

DOI: https://doi.org/10.1137/S0036141099365753



Originally published at:
Kappeler, T; Mityagin, B (2001). Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator. SIAM Journal on Mathematical Analysis, 33(1):113-152.
DOI: https://doi.org/10.1137/S0036141099365753

SIAM J. MATH. ANAL.
Vol. 33, No. 1, pp. 113–152


# ESTIMATES FOR PERIODIC AND DIRICHLET EIGENVALUES OF THE SCHRÖDINGER OPERATOR*

T. KAPPELER† AND B. MITYAGIN‡

**Abstract.** Consider the Schrödinger equation $-y'' + Vy = \lambda y$ for a complex-valued potential $V$ of period 1 in the weighted Sobolev space $H^w$ of 2-periodic functions $f : \mathbb{R} \to \mathbb{C}$,

$$H^w \equiv H^w_{\mathbb{C}} := \left\{ f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{i\pi kx} \mid \|f\|_w < \infty \right\},$$

where

$$\|f\|_w := \left( 2 \sum_k w(k)^2 \; |\hat{f}(k)|^2 \right)^{1/2}$$

and $w = (w(k))_{k \in \mathbb{Z}}$ denotes a symmetric, submultiplicative weight sequence. Denote by $\lambda_n = \lambda_n(V)$ $(n \geq 0)$ the periodic eigenvalues of $-\frac{d^2}{dx^2} + V$ when considered on the interval $[0, 2]$, listed in such a way that $\lambda_{2n}, \lambda_{2n-1} = n^2\pi^2 + 0(1)$, and denote by $\mu_n = \mu_n(V)$ $(n \geq 1)$ the Dirichlet eigenvalues of $-\frac{d^2}{dx^2} + V$ considered on $[0, 1]$, listed in such a way that $\mu_n = n^2\pi^2 + 0(1)$.

THEOREM. *There exist (absolute) constants $K_1, K_2 > 0$, so that for any 1-periodic potential $V$ in $H^w$,*

$$\sum_{n \geq N} w(2n)^2 |\lambda_{2n} - \lambda_{2n-1}|^2 \leq K_1 (1 + \|V\|_w)^{K_2}$$

*and*

$$\sum_{n \geq N} w(2n)^2 |\mu_n - \lambda_{2n}|^2 \leq K_1 (1 + \|V\|_w)^{K_2},$$

*where $N := K_1(1 + \|V\|_w)^2$.*

**Key words.** Schrödinger operators, periodic and Dirichlet eigenvalues, estimates on gap lengths

**AMS subject classifications.** 58F19, 58F07, 35Q35

**PII.** S0036141099365753

## 1. Introduction.

**1.1. Summary of the results.** Consider the Schrödinger equation on the interval $[0, 2]$,

$$-y'' + Vy = \lambda y, \tag{1.1}$$

where $V$ is a complex-valued periodic potential of *period* 1 in the weighted Sobolev space of 2-periodic functions,

$$H^w \equiv H^w_{\mathbb{C}} := \left\{ f(x) = \sum_k \hat{f}(k) e^{i\pi kx} \mid \|f\|_w < \infty \right\}$$

*Received by the editors November 29, 1999; accepted for publication (in revised form) November 28, 2000; published electronically April 24, 2001.
http://www.siam.org/journals/sima/33-1/36575.html
†Institut für Mathematik, Universität Zürich, 8057 Zürich, Switzerland (tk@math.unizh.ch).
‡Department of Mathematics, Ohio State University, Columbus, OH 43210 (borismit@math.ohio-state.edu).

with

$$\|f\|_w := \left(2 \sum_k w(k)^2 \; |\hat{f}(k)|^2\right)^{1/2}$$

and $w = (w(k))_{k\in\mathbb{Z}}$ with $w(k) \geq 1 \; \forall k \in \mathbb{Z}$ is a symmetric weight ($w(k) = w(-k) \; \forall k \in \mathbb{Z}$) which is submultiplicative,

$$w(k+j) \leq w(k)w(j) \; \forall k, j \in \mathbb{Z}.$$

As an example of a submultiplicative weight we mention the Abel–Sobolev weight $w_{a,b}(k) := (1+|k|)^a e^{b|k|}$ with $a \geq 0$ and $b \geq 0$. An element $f \in H^{w_{a,b}}$ can be viewed as a complex-valued function $F(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{i\pi kz}$, $z = x+iy$, analytic in the strip $|y| < \frac{b}{\pi}$ and such that $F(x + i\frac{b}{\pi})$ as well as $F(x - i\frac{b}{\pi})$ are in the Sobolev space $H^a$, defined by the weight $w(k) := (1+|k|)^a$. More generally, $w(k) := (1+|k|)^a e^{b|k|^\alpha}$ is a submultiplicative weight for $0 \leq \alpha \leq 1, a \geq 0$ and $b \geq 0$; a function $f \in H^w$ is a function of Gevrey class.

The spectrum $spec_{Per}(L)$ of the operator $L := -\frac{d^2}{dx^2} + V$, when considered on the interval $[0,2]$ and with periodic boundary conditions, is discrete and is a sequence $\lambda_n = \lambda_n(V)$ $(n \geq 0)$ with the property that $\mathrm{Re}\lambda_n \to +\infty$ for $n \to \infty$. Here, the eigenvalues $\lambda_n$ are enumerated with their algebraic multiplicities and ordered so that

$$\mathrm{Re}\lambda_n < \mathrm{Re}\lambda_{n+1} \qquad \text{or} \qquad \mathrm{Re}\lambda_n = \mathrm{Re}\lambda_{n+1} \text{ and } \mathrm{Im}\lambda_n \leq \mathrm{Im}\lambda_{n+1}.$$

Notice that adding a constant to the potential $V$ results in a shift of the eigenvalues by the same constant. Hence we restrict ourselves—without loss of generality—to potentials $V$ of mean zero and introduce the subspace $H_0^w \subseteq H^w$,

$$H_0^w := \left\{ f \in H_{\mathbb{C}}^w \;\middle|\; \int_0^2 f(x)dx = 0 \right\}.$$

For the weight $w \equiv 1$, the spaces $H_0^w$ and $H^w$ are also denoted by $L_0^2$ and $L^2$, respectively. For $n$ sufficiently large (cf. Lemma 1.4 in section 1.2 for a reminder), the eigenvalues come in pairs $\{\lambda_{2n}, \lambda_{2n-1}\}$, i.e., $\lambda_{2n}$ and $\lambda_{2n-1}$ are close to each other and separated from the rest of $spec_{per}(L)$ by a distance of size $n$.

In section 2 of this paper, we prove the following theorem.

THEOREM 1.1. *There exist (absolute) constants $K_1, K_2 > 0$ so that for any 1-periodic potential $V$ in $H_0^w$*

$$\sum_{n\geq N} w(2n)^2 \; |\lambda_{2n} - \lambda_{2n-1}|^2 \leq K_1(1 + \|V\|_w)^{K_2}, \tag{1.2}$$

*where* $N := K_1(1+\|V\|_w)^2$.

(See Proposition 2.16 and section 2.8 for details.)

In the next theorem we state the main two terms in the asymptotics of the sequence of gap lengths, $\gamma_n := \lambda_{2n} - \lambda_{2n-1}$ as $n \to \infty$. For this purpose introduce

$$\rho(n) := \hat{V}(2n) + \frac{1}{\pi^2} \sum_j \frac{\hat{V}(n-j)}{n-j} \frac{\hat{V}(n+j)}{n+j}. \tag{1.3}$$

Notice that the last term in (1.3) is a convolution and well defined as $\hat{V}(0) = 0$.

THEOREM 1.2. *There exist (absolute) constants $K_3, K_4 > 0$ so that for any 1-periodic potential $V$ in $H_0^w$*

$$\sum_{n \geq N} (1+|n|)^2 w(2n)^2 \min_{\pm} \mid (\lambda_{2n} - \lambda_{2n-1}) \pm 2\sqrt{\rho(n)\rho(-n)} \mid^2 \leq K_3(1+\|V\|_w)^{K_4}, \tag{1.4}$$

*where* $N := K_3(1+\|V\|_w)^2$.

(See Theorem 2.20 and section 2.9 for further details.)

In our previous paper [8], we obtained estimate (1.2) and a weaker form of estimate (1.4) for the Abel–Sobolev weights

$$w_{a,b} := (1+|k|)^a e^{b|k|}, \quad (a \geq 0, b \geq 0),$$

using a Fourier approach. By a refined analysis we obtain in section 2 of the present paper estimates (1.2) for general submultiplicative weights and a two-terms asymptotic (1.3)–(1.4) for the gap lengths. It turns out that submultiplicative weights provide the right setup for applications to a KAM theorem for the Korteweg–deVries equation (cf. [2]), as will be shown in a subsequent paper. Further, we present in the present paper an analysis of the Riesz spaces together with estimates for the Dirichlet eigenvalues. Let us explain this in more detail.

In section 3, we analyze the Riesz spaces $E_n$ ($n$ sufficiently large), i.e., the images of the Riesz projectors defined by a circle of appropriate size around $n^2\pi^2$ as contour and the operator $L := \frac{d^2}{dx^2} + V$, considered on $[0,1]$ with periodic (for $n$ even) or antiperiodic (for $n$ odd) boundary conditions (cf. (1.16)). We study the structure of $L$ by computing the matrix representation of the restriction of $L - \lambda_{2n}$ to $E_n$ with respect to an orthonormal basis $f_n, \varphi_n$, where $f_n$ is a periodic or antiperiodic eigenfunction in $E_n$. Moreover, we estimate the entries of this matrix which will be important for estimates of the Dirichlet eigenvalues (cf. Theorem 3.5 and Proposition 3.6).

In section 4 we obtain estimates for the Dirichlet eigenvalues $\mu_n(V)$ ($n \geq 1$) of the operator $-\frac{d^2}{dx^2} + V$, considered on the interval $[0,1]$. The eigenvalues $\mu_n \equiv \mu_n(V)$ are ordered in such a way that

$$\mathrm{Re}\mu_n < \mathrm{Re}\mu_{n+1} \qquad \text{or} \qquad \mathrm{Re}\mu_n = \mathrm{Re}\mu_{n+1} \text{ and } \mathrm{Im}\mu_n \leq \mathrm{Im}\mu_{n+1}. \tag{1.5}$$

THEOREM 1.3. *There exist (absolute) constants $K_5, K_6 > 0$ so that for any 1-periodic potential $V$ in $H_0^w$*

$$\sum_{n \geq N} w(2n)^2 |\mu_n - \lambda_{2n}|^2 \leq K_5(1+\|V\|_w)^{K_6}, \tag{1.6}$$

*where* $N := K_5(1+\|V\|_w)^{K_6}$.

It turns out that by the methods used to prove Theorem 1.3, one can obtain similar results for the eigenvalues of $L_{bc}$, where $L_{bc}$ is the operator $L$ with boundary conditions $bc$ from a special class $\mathcal{B}$. In section 5, this class is defined and the spectrum of the operators $L_{bc}$ is analyzed.

It is well known that the decay of the gap lengths $\gamma_n := \lambda_{2n} - \lambda_{2n-1}$, associated to $spec_{Per}(L)$, depends on the smoothness properties of $V$ (cf., e.g., [7], [13], [20]). In particular Marčenko [13] obtains polynomial decay of the gap lengths in terms of the Sobolev class of the potential and Trubowitz [20] proves exponential decay for real analytic potentials. Conversely, the question of smoothness of an $L_2$-potential in terms of the decay of the gap lengths has been addressed as well, mainly for real-valued potentials (cf. [13], [15], [20]) but more recently also for complex-valued potentials. It turns out that for complex-valued potentials, the decay of the gap lengths does not suffice to determine the smoothness: Sansuc and Tkachenko [19] proved that a periodic complex-valued potential $V \in L_0^2$ belongs to the Sobolev space $H_0^N$ iff the following two conditions are satisfied:

$$\sum_{n\geq 1}(1+|n|)^{2N}\,|\lambda_{2n}-\lambda_{2n-1}|^2<\infty\ ;\quad \sum_{n\geq 1}(1+|n|)^{2N}\,|\mu_n-\lambda_{2n}|^2<\infty,$$

where, as above, $(\mu_n)_{n\geq 1}$ denote Dirichlet eigenvalues.

The condition of the weight sequence $(w(n))_{n\in\mathbb{Z}}$ to be submultiplicative could be seen as purely technical and convenient in the proofs of the inequalities stated in the theorems above, but it may be too restrictive for results like Theorem 1.1. Moreover, the submultiplicativity implies that

$$\lim_{n\to\infty}\frac{\log w(n)}{n}=\omega_*<\infty.$$

Thus, for $\omega>\omega_*$,

$$w(n)\leq C_\omega e^{\omega|n|}\quad \forall n\in\mathbb{Z} \tag{1.7}$$

for some constant $C_\omega>0$ and $w(n)$ cannot grow faster than an exponential function. Notice, however, that the slightest violation of the growth restriction (1.7) gives a weight sequence which does not have the property stated by Theorem 1.1. This follows from Harrell's and Grigis's analysis of the gap lengths for (real) polynomial potentials. If $V$ is a Mathieu potential

$$V(x)=t\cos(2\cdot 2\pi x)\quad 0\leq x\leq 1, \tag{1.8}$$

then Harrell [6] (cf. [1]) proved that the gap lengths $\gamma_n$ satisfy the asymptotic estimates (cf. [4, formula (1.8)])

$$\gamma_n=\frac{t^n}{8^{n-1}((n-1)!)^2}\left(1+0\left(\frac{1}{n^2}\right)\right),$$

and therefore, for some $a$, depending on $t$,

$$\gamma_n>e^{-2n\log n+an}. \tag{1.9}$$

Hence, if $w(n):=e^{b|n|\log|n|}$ with $b>2$, the analogue of Theorem 1.1 does not hold. Indeed, we have

$$\|V\|_w^2=\frac{\pi|t|^2}{2}w(4)^2<\infty,$$

but (compare with (1.2))

$$\sum_{n\geq N}w(2n)^2|\gamma_n|^2\geq\sum_{n\geq N}e^{4bn\log n}e^{-4n\log n+2an}=\infty.$$

A more refined analysis due to Grigis (see [4, Theorem 0.2]) shows that the above weight is bad with any $b>0$, i.e., does not have the property stated by Theorem 1.1.

**1.2. Preliminaries.** General references on Schrödinger operators on the interval and Hill's operator can be found, e.g., in [11], [12], [18].

In this section we put together some well-known spectral properties of the operator $L := -\frac{d^2}{dx^2} + V$ in a form convenient for our further analysis. The following three lemmas are particular results in the general theory of nonselfadjoint boundary value problems developed by Keldysh [9], [10]. Many details can be found in [14], section 6 of chapter 1, in particular in subsection 6.3 (Lemmas 6.6 and 6.7) and 6.4 (p. 34); cf. also the appendix (pp. 215–219) where the paper [9] is translated into English.

Let us consider Dirichlet boundary conditions, $bc = Dir$, as well as periodic $Per^+$ and antiperiodic $Per^-$ boundary conditions, $bc = Per^{\pm}$, i.e., for functions $y$ in $H^2_{\mathbb{C}}[0,1]$,

$(Dir)$ $\qquad y(0) = 0\ ; \quad y(1) = 0;$

$(Per^+)$ $\qquad y(1) = y(0)\ ; \quad y'(1) = y'(0);$

$(Per^-)$ $\qquad y(1) = -y(0)\ ; \quad y'(1) = -y'(1).$

For $V \in L^2_{\mathbb{C}}[0,1]$ with $\int_0^1 V(x)dx = 0$ introduce the operator $L := D^2 + V$, where $D = \frac{1}{i}\frac{d}{dx}$. Given one of the above boundary conditions $bc$, denote by $L_{bc}$ the closed operator in $L^2_{\mathbb{C}}[0,1]$ with domain $dom(L_{bc}) := \{f \in H^2_{\mathbb{C}}([0,1]) | f \text{ satisfies } bc\}$. Let $spec_{bc}(L) \equiv spec(L_{bc})$ be the spectrum of $L_{bc}$. For the potential $V \equiv 0$, i.e., $L = D^2$, $spec_{bc}(D^2)$ can be given explicitly,

$$spec_{Dir}(D^2) = \{k^2\pi^2 | k \geq 1\}, \tag{1.10}$$

$$spec_{Per^+}(D^2) = \{0\} \cup \{(2k)^2\pi^2, (2k)^2\pi^2 | k \geq 1\}, \tag{1.11}$$

$$spec_{Per^-}(D^2) = \{(2k-1)^2\pi^2, (2k-1)^2\pi^2 | k \geq 1\}. \tag{1.12}$$

For $r > 0$ and $k \in \mathbb{Z}_{\geq 0}$, let $\mathcal{D}(k) \equiv \mathcal{D}_r(k)$ be the open disc in $\mathbb{C}$ with center $k^2\pi^2$ and radius $r$

$$\mathcal{D}(k) := \{z \in \mathbb{C} \mid \quad |z - k^2\pi^2| < r\}$$

and, for $r_1, r_2 > 0, \mathcal{R} \equiv \mathcal{R}_{r_1,r_2}$ the open rectangle in $\mathbb{C}$

$$\mathcal{R} := \{x + iy \mid \quad -r_1 < x < r_2;\ |y| < r_2\}.$$

Denote by $\|V\|$ the $L^2$-norm of $V \in L^2_{\mathbb{C}}[0,2], \|V\| = (2\sum_k |\hat{V}(k)|^2)^{1/2}$.

Lemma 1.4. *There exist absolute constants $K_7 \geq 1$ and $K_8 \geq 1$ so that, for any given $M \geq 1$, boundary condition $bc \in \{Dir, Per^{\pm}\}$, $N \geq 2K_8(M+1)$, and 1-periodic potential $V \in L^2_{\mathbb{C}}[0,2]$ with $\|V\| \leq M$, the following holds:*

$$spec(L_{bc}) \subset \mathcal{R} \cup \bigcup_{k=N+1}^{\infty} \mathcal{D}(k), \tag{1.13}$$

*where $\mathcal{D}(k) \equiv \mathcal{D}_r(k)$ with $r := K_8(M+1)$ and $\mathcal{R} = \mathcal{R}_{r_1,r_2}$ with $r_1 = K_7(1+M)^{4/3}$ and $r_2 = (N^2+N)\pi^2$.*

We point out that $spec_{Per^+}(D^2) \cup spec_{Per^-}(D^2)$ (cf. (1.11) and (1.12)) is the spectrum $spec_{Per}(D^2)$ of the operator $D^2$ on $[0,2]$ with periodic boundary conditions. Obviously, for any 1-periodic potential $V$,

$$spec_{Per^+}(L) \cup spec_{Per^-}(L) \subseteq spec_{Per}(L).$$

For a real-valued potential the converse inclusion

$$spec_{Per}(L) \subseteq spec_{Per^+}(L) \cup spec_{Per^-}(L) \tag{1.14}$$

also holds, as one can see from an elementary application of Floquet theory. More generally, by a simple counting argument, Lemma 1.4 implies that (1.14) holds for complex-valued potentials.

The periodic eigenvalues of $L$ on $[0,2]$ have been denoted by $(\lambda_n)_{n\geq 0}$ (cf. (1.7)). According to Lemma 1.4, the eigenvalues $\lambda_{2n-1}$ and $\lambda_{2n}$ are close to $n^2\pi^2$ for $n$ sufficiently large. At certain occasions (cf., e.g., section 2.8), one of the two eigenvalues, either $\lambda_{2n}$ or $\lambda_{2n-1}$, will satisfy a certain property, but it will not be possible to decide which of the two. For such a situation, it is convenient to introduce $\lambda_n^+, \lambda_n^-$ as a different notation for the eigenvalues $\lambda_{2n}, \lambda_{2n-1}$,

$$\{\lambda_n^+, \lambda_n^-\} = \{\lambda_{2n}, \lambda_{2n-1}\}.$$

By Lemma 1.4, it follows that the Riesz projectors $P_* \equiv P_{*;bc}$ and $P_k \equiv P_{k;bc}$ are well defined for $\|V\| \leq M$,

$$P_* := \frac{1}{2\pi i}\int_{\partial\mathcal{R}} (z - L_{bc})^{-1} dz, \tag{1.15}$$

$$P_k := \frac{1}{2\pi i}\int_{\partial\mathcal{D}(k)} (z - L_{bc})^{-1} dz, \quad (k \geq N+1), \tag{1.16}$$

where the contours $\partial\mathcal{R}$ and $\partial\mathcal{D}(k)$ are counterclockwise oriented. Denote by $\|T\|_{\mathcal{L}(L^2)}$ the operator norm of a bounded linear operator $T : L^2_{\mathbb{C}}[0,1] \to L^2_{\mathbb{C}}[0,1]$.

Lemma 1.5. *There exist absolute constants $K_9$ and $K_{10}$ so that under the same assumptions as in Lemma* 1.4*,*

$$\|P_*\|_{\mathcal{L}(L^2)} \leq K_9 \log(2+M), \tag{1.17}$$

$$\|P_k\|_{\mathcal{L}(L^2)} \leq K_{10}, \quad (k \geq N+1). \tag{1.18}$$

*Further, for any $f \in L^2_{\mathbb{C}}[0,1]$,*

$$f = P_* f + \sum_{k=N+1}^{\infty} P_k f, \tag{1.19}$$

*where the series* (1.19) *converges in $L^2$.*

Lemma 1.6. *There exists an absolute constant $K_{11} \geq 1$ so that under the same assumptions as in Lemma* 1.4*,*

$$\|(\lambda - L_{bc})^{-1}(Id - P_k)\|_{\mathcal{L}(L^2)} \leq K_{11}\frac{1}{k} \quad \forall \lambda \in \mathcal{D}_r(k), \quad \forall k \geq N+1. \tag{1.20}$$

## 2. Periodic eigenvalues.

**2.1. Fourier block decomposition.** Denote by $L$ the Schrödinger operator $L := D^2 + V$, $D = \frac{1}{i}\frac{d}{dx}$ with a complex-valued potential $V \in H_0^w$ of period 1, considered as an unbounded operator on $L^2_{\mathbb{C}}[0,2]$, with periodic boundary conditions. For $V = 0$, the spectrum is discrete: $0, \pi^2, \pi^2, (2\pi)^2, (2\pi)^2, \dots$ ; i.e., the eigenvalues $k^2\pi^2$ are double for $k \geq 1$ and the eigenvalues $(n+1)^2\pi^2$ and $n^2\pi^2$ are $(2n+1)\pi^2$ apart. Further, for $n \geq 1$, $e^{in\pi x}, e^{-in\pi x}$ is a basis of the eigenspace corresponding to the eigenvalue $n^2\pi^2$. Viewing the potential $V$ as a perturbation of $D^2$, it follows that for $n$ sufficiently large, $L$ has a pair of eigenvalues near $n^2\pi^2$, isolated from the remaining spectrum of $L$. Our aim is to obtain an estimate for the distance between the two eigenvalues and to compare eigenfunctions and eigenvalues with the corresponding ones for $V = 0$. Notice, however, that $L$ might have double eigenvalues of geometric multiplicity 1 as $V$ is complex-valued.

The Fourier series decomposition leads to an isometric isomorphism $\mathcal{F}$ between $L^2_{\mathbb{C}}[0,2]$ and $\ell^2(\mathbb{Z})$ with $\mathcal{F}(e^{i\pi kx}) = e_k$, $(e_k)_{k\in\mathbb{Z}}$ being the standard basis in $\ell^2(\mathbb{Z})$. Decompose $\hat{L} = \mathcal{F}L\mathcal{F}^{-1}$ with respect to the orthogonal sum $\ell^2(\mathbb{Z}) = \mathbb{C}e_{-n} \oplus \mathbb{C}e_n \oplus \ell^2(\mathbb{Z}\backslash\{\pm n\})$. To express $\hat{L}$, introduce the involution operator $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$(Ja)(k) := a(-k) \; (k \in \mathbb{Z})$$

and the shift operator $\mathcal{S} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$(\mathcal{S}a)(k) := a(k+1) \; (k \in \mathbb{Z}).$$

$\mathcal{S}^n = \mathcal{S} \circ \cdots \circ \mathcal{S}$ denotes the $n$*th iterate of $\mathcal{S}$. For any subset $K \subset \mathbb{Z}$, the restriction of $\mathcal{S}$ on $\ell^2(K)$ with values in $\ell^2(\mathcal{S}(K))$ is denoted by $\mathcal{S}$ as well. This leads to the block decomposition of $\hat{L} - \lambda, \lambda = n^2\pi^2 + z$,

$$\hat{L} - (n^2\pi^2 + z) = \begin{pmatrix} -z & \hat{V}(-2n) & (\mathcal{S}^n J\hat{V})^t_{\mathbb{Z}(n)} \\ \hat{V}(2n) & -z & (\mathcal{S}^{-n} J\hat{V})^t_{\mathbb{Z}(n)} \\ (\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)} & (\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)} & A_n - z \end{pmatrix}, \tag{2.1}$$

where $\mathbb{Z}(n) := \mathbb{Z}\backslash\{\pm n\}$, the superscript $t$ denotes the transpose, and $A_n : \ell^2(\mathbb{Z}\backslash\{\pm n\}) \to \ell^2(\mathbb{Z}\backslash\{\pm n\})$ is the linear operator with matrix representation

$$A_n(j,k) = \pi^2(k^2 - n^2)\delta_{jk} + \hat{V}(j-k), \quad (j,k \in \mathbb{Z}(n)).$$

The (possibly) complex number $\lambda = n^2\pi^2 + z$ is a periodic eigenvalue of $L$ if there exists a 2-periodic function $f \in H^2_{\mathbb{C}}([0,2])$ such that $(L - \lambda)f = 0$. With

$$x^f := \hat{f}(-n), \quad y^f := \hat{f}(n), \quad F := (\hat{f}(k))_{\mathbb{Z}(n)},$$

the equation $(L - \lambda)f = 0$, or its equivalent $(\hat{L} - \lambda)\hat{f} = 0$, leads to the following homogeneous system of equations:

$$-zx^f + \hat{V}(-2n)y^f + [\mathcal{S}^n J\hat{V}, F]_{\mathbb{Z}(n)} = 0, \tag{2.2}$$

$$\hat{V}(2n)x^f - zy^f + [\mathcal{S}^{-n} J\hat{V}, F]_{\mathbb{Z}(n)} = 0, \tag{2.3}$$

$$(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}x^f + (\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)}y^f + (A_n - z)F = 0, \tag{2.4}$$

where $[a,b]_K = \sum_{k\in K} a(k)b(k)$ (no complex conjugation). Equation (2.4) will be referred to as *the external equation*. The system of equations (2.2)–(2.4) is analyzed as follows. First we solve the external equation (2.4) for $F$, regarding $x^f, y^f$, and $z$ as parameters. The solution $F$ of (2.4) is then substituted into the equations (2.2)–(2.3). This leads to a linear homogeneous system of two equations for the unknowns $x^f, y^f$ with parameter $z$. The determinant of this system vanishes iff $\lambda = n^2\pi^2 + z$ is an eigenvalue of $L$. In section 3.3 we will also consider the inhomogneous version of the system (2.2)–(2.4) in order to obtain, among other results, an orthonormal basis of the root space of a double eigenvalue of $L$ of geometric multiplicity 1.

**2.2. Analysis of the external equation.** To analyze the operator $(A_n - z) : \ell^2(\mathbb{Z}(n)) \to \ell^2(\mathbb{Z}(n))$, we write $A_n = D_n + B_n$, where $D_n$ is the diagonal part of $A_n$ (recall that $\hat{V}(0) = 0$),

$$D_n(k,j) := \pi^2(k^2 - n^2)\delta_{kj}, \quad (k, j \in \mathbb{Z}(n) = \mathbb{Z}\backslash\{\pm n\}). \tag{2.5}$$

Notice that $D_n$ is invertible and that $B_n$ has matrix elements

$$B_n(k,j) = \hat{V}(k-j), \quad (k,j \in \mathbb{Z}(n)).$$

Write

$$A_n - z = D_n - (z - B_n) = (Id - T_n)D_n; \quad T_n := (z - B_n)D_n^{-1}, \tag{2.6}$$

where $T_n$ is an operator on $\ell^2(\mathbb{Z}(n))$ with matrix elements

$$(z\delta_{kj} - \hat{V}(k-j))\frac{1}{\pi^2(j-n^2)}.$$

Further, denote by $\|V\|$ the norm of $V$ in $L^2_{\mathbb{C}}([0,2]), \|V\| = (2\sum_k |\hat{V}(k)|^2)^{1/2}$.

Lemma 2.1. (i) *For* $n \geq 1$,

$$\|D_n^{-1}\| \;\leq\; \frac{1}{\pi^2}\frac{1}{n}, \tag{2.7}$$

$$\|T_n\| \;\leq\; \frac{1}{3n}(|z| + \|V\|); \tag{2.8}$$

(ii) *for* $n \geq 1$ *and* $z \in \mathbb{C}$ *with* $|z| + \|V\| \;\leq\; n$,

$$\|(A_n - z)^{-1}\| \;\leq\; \frac{2}{\pi^2}\frac{1}{n}. \tag{2.9}$$

*Proof.* (i) (2.7) follows from (2.5). Concerning (2.8) we prove $\|T_n\|_{HS} \leq \frac{1}{3n}(|z| + \|V\|)$ with $\|T_n\|_{HS}$ denoting the Hilbert–Schmidt norm of $T$ (which leads to a stronger version of (2.8) as $\|T_n\| \leq \|T_n\|_{HS}$):

$$\begin{aligned}\|T_n\|^2_{HS} &= \sum_{j,k\in\mathbb{Z}(n)} \frac{|z\delta_{kj} - \hat{V}(k-j)|^2}{|\pi^2(k^2-n^2)|^2} \\ &\leq \sum_{k\in\mathbb{Z}(n)} \frac{1}{\pi^4}\frac{2|z|^2 + 2\|\hat{V}\|^2}{(k-n)^2(k+n)^2}.\end{aligned}$$

As

$$\sum_{k\neq\pm n}\frac{1}{(k-n)^2(k+n)^2}=\frac{1}{6}\left(\frac{\pi}{n}\right)^2-\frac{3}{8}\frac{1}{n^4}, \tag{2.10}$$

we conclude that $\|T_n\|_{HS}\leq\frac{1}{3n}(|z|+\|V\|)$.

(ii) If $\|T_n\|\leq\frac{1}{2}$, $(A_n-z)$ is invertible (cf. (2.6)) and

$$\|(A_n-z)^{-1}\| \ \leq\ 2\|D_n^{-1}\| \ \leq\ \frac{2}{\pi^2}\frac{1}{n}.$$

In view of (2.8), $\|T_n\|\leq 1/2$ for $|z|+\parallel V\parallel\leq n$. ◻

As an immediate consequence of Lemma 2.1, one obtains the following proposition.

Proposition 2.2. *Let $n\geq 1$ and $z\in\mathbb{C}$ satisfy $|z|+\parallel V\parallel\leq n$. Then, for any choice of $x^f,y^f$ in $\mathbb{C}$, (2.4) has a unique solution $F$*

$$F=(z-A_n)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}x^f+(z-A_n)^{-1}(\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)}y^f. \tag{2.11}$$

Substituting the solution $F$, given by (2.11), into (2.2)–(2.3), one gets

$$\begin{pmatrix} -z+\alpha(-n,z) & \hat{V}(-2n)+\beta(-n,z)\\ \hat{V}(2n)+\beta(n,z) & -z+\alpha(n,z)\end{pmatrix}\begin{pmatrix} x^f\\ y^f\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \tag{2.12}$$

where, for $n\in\mathbb{Z}\backslash\{0\}$, satisfying $|z|+\|V\|\leq n$, we define, with $A_n:=A_{|n|}$,

$$\alpha(n,z):=[\mathcal{S}^{-n}J\hat{V},(z-A_n)^{-1}(\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}, \tag{2.13}$$

$$\beta(n,z):=[\mathcal{S}^{-n}J\hat{V},(z-A_n)^{-1}(\mathcal{S}^{n}\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}. \tag{2.14}$$

In the following sections, the coefficients $\alpha(n,z)$ and $\beta(n,z)$ will be analyzed. Often, we will write $[\cdot,\cdot]$, for $[\cdot,\cdot]_{\mathbb{Z}(n)}$.

**2.3. Identity for $\boldsymbol{\alpha(n,z)}$.** Throughout this and the following section, we assume that $n\geq 1$ and $z\in\mathbb{C}$ are such that $|z|+\|V\|\leq n$. To simplify notation we drop the subindex $n$ in $A_n,B_n,D_n$, and $T_n$ with the understanding that $n$ is fixed in this subsection. Denote by $(z-A)^t$ the transpose of $z-A$, $(z-A)^t(j,k):=(z-A)(k,j)(\forall k,j\in\mathbb{Z}(n))$.

Lemma 2.3. (i) $J\mathcal{S}^n=\mathcal{S}^{-n}J$;

(ii) $(z-A)=J(z-A)^tJ$.

*Proof.* Part (i) is verified in a straightforward way. Regarding (ii), it is to prove that for $k,j\in\mathbb{Z}(n)$,

$$(z-A)(k,j)=(z-A)(-j,-k). \tag{2.15}$$

The identity (2.15) follows from the definition

$$(z-A)(k,j)=z\delta_{kj}-\pi^2(k^2-n^2)\delta_{kj}-\hat{V}(k-j)$$

and the identities

$$\delta_{kj}=\delta_{(-j)(-k)};\quad \hat{V}(k-j)=\hat{V}(-j-(-k)).\qquad ◻$$

We obtain the following identity for $\alpha(n,z)$.

LEMMA 2.4. $\alpha(n,z) = \alpha(-n,z)$.

*Proof.* With $(\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)} = \mathcal{S}^{-n}\hat{V}_{\mathbb{Z}\setminus\{0,2n\}}$ and $(z-A)^{-1} = (J(z-A)^tJ)^{-1} = J((z-A)^{-1})^tJ$ (Lemma 2.3) it follows that

$$\begin{aligned}
\alpha(n,z) &= [\mathcal{S}^{-n}J\hat{V}, (J(z-A)^tJ)^{-1}\mathcal{S}^{-n}\hat{V}_{\mathbb{Z}\setminus\{0,2n\}}]_{\mathbb{Z}(n)} \\
&= [J\mathcal{S}^{n}\hat{V}, J((z-A)^{-1})^tJ\mathcal{S}^{-n}\hat{V}_{\mathbb{Z}\setminus\{0,2n\}}]_{\mathbb{Z}(n)} \\
&= [(z-A)^{-1}\mathcal{S}^{n}\hat{V}, J\mathcal{S}^{-n}\hat{V}_{\mathbb{Z}\setminus\{0,2n\}}]_{\mathbb{Z}(n)} \\
&= [\mathcal{S}^{n}J\hat{V}, (z-A)^{-1}\mathcal{S}^{n}\hat{V}]_{\mathbb{Z}(n)} \\
&= \alpha(-n,z). \qquad \square
\end{aligned}$$

As a consequence of Lemma 2.4, the vanishing of the determinant of the $2\times 2$ matrix in (2.12) leads to the following equation for $z$:

$$(z-\alpha(n,z))^2 - \left(\hat{V}(2n)+\beta(n,z)\right)\left(\hat{V}(-2n)+\beta(-n,z)\right) = 0. \tag{2.16}$$

Equation (2.16) is solved in two steps: for $\zeta$ given, we first solve the following equation, referred to as the $z$-equation, for $z$:

$$z = \alpha(n,z)+\zeta. \tag{2.17}$$

Substituting the solution $z = z(\zeta)$ of (2.17) into (2.16), we obtain the following equation for $\zeta$, referred to as the $\zeta$-equation:

$$\zeta^2 - \left(\hat{V}(2n)+\beta(n,z(\zeta))\right)\left(\hat{V}(-2n)+\beta(-n,z(\zeta))\right) = 0. \tag{2.18}$$

In the next four sections, (2.17) and (2.18) will be analyzed.

**2.4. Estimates of $\alpha(n,z)$ and the $z$-equation (2.17).** In this section we solve (2.17), using the contractive mapping principle. For this purpose we need the following lemma.

LEMMA 2.5. *For $n \geq 1$ and $z \in \mathbb{C}$ satisfying $|z| + \|V\| \leq n$,*

(i) $|\alpha(n,z)| \leq \|V\|^2/3n$;

(ii) $|\frac{d}{dz}\alpha(n,z)| \leq \|V\|^2/9n^2$.

*Proof.* (i) By the definition (2.13) and Lemma 2.1,

$$|\alpha(n,z)| \leq \|V\| \|(z-A)^{-1}\| \, \|V\| \leq \frac{2}{\pi^2}\frac{1}{n}\|V\|^2.$$

(ii) Notice that

$$\frac{d}{dz}\alpha(n,z) = [\mathcal{S}^{-n}J\hat{V}, -(z-A)^{-2}\mathcal{S}^{-n}\hat{V}]_{\mathbb{Z}(n)},$$

and therefore,

$$\left|\frac{d}{dz}\alpha(n,z)\right| \leq \|V\|\|(z-A)^{-1}\|^2\|V\| \leq \frac{4}{\pi^4}\frac{1}{n^2}\|V\|^2. \qquad \square$$

Denote by $\mathcal{D}_M \equiv \mathcal{D}_M(0)$ the disc $\{z \in \mathbb{C} \mid |z| < M\}$ and denote by $\overline{\mathcal{D}_M}$ its closure.

PROPOSITION 2.6. *Let $V \in L_0^2$. Then for any $M > 0$ and $n \geq 1$ satisfying $n \geq \|V\| + M$, and for any $\zeta \in \mathcal{D}_{M/2}$, the equation*

$$z = \zeta + \alpha(n, z) \tag{2.19}$$

*has a unique solution $z_n = z_n(\zeta)$ in $\mathcal{D}_M$. The solution $z_n(\zeta)$ depends analytically on $\zeta \in \mathcal{D}_{M/2}$.*

*Proof.* For $z \in \overline{\mathcal{D}_M}$,

$$|z| + \|V\| \leq M + \|V\| \leq n,$$

and thus, by Lemma 2.5, $|\alpha(n, z)| \leq M/3$. It follows that for $\zeta \in \mathcal{D}_{M/2}, z \in \overline{\mathcal{D}_M}$

$$|\zeta| + |\alpha(n, z)| \leq M/2 + M/3 < M.$$

Thus, for $\zeta \in \mathcal{D}_{M/2}$, $g(z) := \zeta + \alpha(n, z)$ defines a map on $\overline{\mathcal{D}_M}$ into $\overline{\mathcal{D}_M}$. Furthermore, $g$ is a contraction, as for any $z_1, z_2 \in \mathcal{D}_M$

$$|g(z_1) - g(z_2)| < \frac{1}{9}|z_1 - z_2|,$$

where we used that by Lemma 2.5

$$\sup_{|z| \leq M} \left| \frac{d}{dz}\alpha(n, z) \right| \ \leq \ \frac{1}{9}\frac{1}{n^2}\|V\|^2 \leq \frac{1}{9}.$$

Hence there exists a fixed point $z = z(\zeta)$ of $g$ with $|z| \leq M, z$ with

$$\frac{dz}{d\zeta} = \left(1 - \frac{d\alpha}{dz}(n, z(\zeta))\right)^{-1} \quad \forall |\zeta| < M/2. \qquad \square$$

In a next step, we analyze (2.18). To obtain estimates for the coefficient $\beta(n, z)$ we need to establish bounds for the norm of the operator $T = (z - B)D^{-1}$ introduced in section 2.2, viewed as an operator on a weighted $\ell^2$-space.

**2.5. Estimates of norms of $T_n$.** Recall that, with $n$ arbitrary and fixed, $T \equiv T_n = (z - B)D^{-1} : \ell^2(\mathbb{Z}(n)) \to \ell^2(\mathbb{Z}(n))$ is a bounded operator (cf. (2.6)). If $V \in H_0^w, T$ can also be viewed as an element in $\mathcal{L}(\ell^2_{\mathcal{S}^{\pm n}w}(\mathbb{Z}(n)))$, where $\mathcal{S}^{\pm n}w$ is the shifted weight

$$(\mathcal{S}^{\pm n}w)(j) := w(\pm n + j). \tag{2.20}$$

Denote by $W_\pm : \ell^2_{\mathcal{S}^{\pm n}w}(\mathbb{Z}(n)) \to \ell^2(\mathbb{Z}(n))$ the diagonal operator given by

$$W_\pm(k, j) = w(k \pm n)\delta_{kj}.$$

Notice that $W_\pm : \ell^2_{\mathcal{S}^{\pm n}w}(\mathbb{Z}(n)) \to \ell^2(\mathbb{Z}(n))$ is an isometry. Therefore,

$$\|T_\pm\|_{\mathcal{L}(\ell^2)} = \|T_n\|_{\mathcal{L}(\ell^2_{\mathcal{S}^{\pm n}w})}, \tag{2.21}$$

where $T_\pm := W_\pm T_n W_\pm^{-1} : \ell^2(\mathbb{Z}(n)) \to \ell^2(\mathbb{Z}(n))$.

LEMMA 2.7. *For $n \geq 1$*

$$\|T_n\|_{\mathcal{L}(\ell^2_{\mathcal{S}^{\pm n}w})} \leq \frac{|z| + \|V\|_w}{3n}.$$

*Proof.* In view of (2.21) it suffices to estimate the Hilbert–Schmidt norm of $T_\pm$ in $\mathcal{L}(\ell^2)$. As $w$ is submultiplicative,

$$\frac{(\mathcal{S}^{\pm n} w)(j)}{(\mathcal{S}^{\pm n} w)(k)} \le w(j-k).$$

In view of (2.21), (2.6), and $\hat{V}(0) = 0$,

$$\begin{aligned}
\|T_\pm\|_{HS}^2 = &\sum_{j,k \neq \pm n} \left| \frac{\mathcal{S}^{\pm n} w(j)}{\mathcal{S}^{\pm n} w(k)} \right|^2 |\hat{V}(j-k)|^2 \frac{1}{\pi^4 |k^2 - n^2|^2} \\
&+ \sum_{k \neq \pm n} |z|^2 \frac{1}{\pi^4 |k^2 - n^2|^2} \\
&\le (\|V\|_w^2 + |z|^2) \sum_{k \neq \pm n} \frac{1}{\pi^4} \frac{1}{(k-n)^2 (k+n)^2} \\
&\le \frac{|z|^2 + \|V\|_w^2}{9} \frac{1}{n^2},
\end{aligned}$$

where in the last inequality, we again use (2.10). This estimate leads to $\|T_\pm\|_{HS} \le \frac{|z| + \|V\|_n}{3n}$. ∎

As an immediate consequence of Lemma 2.7 one obtains the following corollary.

Corollary 2.8. *For $n \ge 1$, and $z \in \mathbb{C}$ with $|z| + \|V\|_w \le n$, $Id - T_n$ is invertible and*

$$\|(Id - T_n)^{-1}\|_{\mathcal{L}(\ell^2_{\mathcal{S}^{\pm n} w})} \le 2.$$

Corollary 2.8 can be used to obtain an estimate of the solution $F$ of the external equation established in Proposition 2.2. According to (2.11),

$$F = x^f F_+ + y^f F_-,$$

where

$$F_\pm = (z - A_n)^{-1} (\mathcal{S}^{\pm n} \hat{V})_{\mathbb{Z}(n)}. \tag{2.22}$$

Corollary 2.9. *For $n \ge 1$ and $z \in \mathbb{C}$ with $|z| + \|V\|_w \le n$,*

$$\|F_\pm\|_{\ell^2_{\mathcal{S}^{\pm n} w}} \le \frac{2}{\pi^2 n} \|V\|_w.$$

*Proof.* By (2.6), $(z - A_n)^{-1} = -D_n^{-1} (Id - T_n)^{-1}$, and by Corollary 2.8

$$\|(Id - T_n)^{-1}\|_{\mathcal{L}(\ell^2_{\mathcal{S}^{\pm n} w})} \le 2.$$

As $D_n$ is the diagonal operator on $\ell^2(\mathbb{Z}(n))$ with coefficients $D_n(k,j) = \pi^2 (k^2 - n^2) \delta_{kj}$, we have

$$\|D_n^{-1}\|_{\mathcal{L}(\ell^2_{\mathcal{S}^{\pm n} w})} \le \frac{1}{\pi^2 n}.$$

Combining these estimates yields the claimed estimate. ∎

**2.6. Estimate for $\beta(n,z)$.** Substitute, for $z \in \mathbb{C}$ satisfying $|z| + \|V\| \leq n$,

$$(z-A)^{-1} = -D^{-1} - D^{-1}T(Id-T)^{-1}$$

into the expression for $\beta(n,z)$ to obtain

$$\begin{aligned}\beta(n,z) &= \beta_1(n) + \beta_2(n,z) \\ &= \beta_1(n) + \beta_2(n,0) + z\int_0^1 \left(\frac{d}{dz}\beta\right)(n,tz)dt,\end{aligned} \tag{2.23}$$

where

$$\beta_1(n) := -[\mathcal{S}^{-n}J\hat{V}, D^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}, \tag{2.24}$$

$$\beta_2(n,z) := -[\mathcal{S}^{-n}J\hat{V}, D^{-1}T(Id-T)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}. \tag{2.25}$$

The term $\beta_1(n)$ is independent of $z$ and

$$\beta_1(n) = \frac{1}{\pi^2}\left(\frac{\hat{V}}{k} * \frac{\hat{V}}{k}\right)(2n), \tag{2.26}$$

or

$$\beta_1(n) = \frac{1}{\pi^2}\sum_{k\neq\pm n}\frac{\hat{V}(n-k)}{n-k}\frac{\hat{V}(n+k)}{n+k}, \tag{2.27}$$

where we use that $-D(k,j) = -\pi^2(k^2-n^2)\delta_{kj} = \pi^2(n-k)(n+k)\delta_{kj}$. In the subsequent lemmas, $\beta_1(n), \beta_2(n,0)$, and $\frac{d}{dz}\beta(n,z)$ are estimated separately. Given the weight $w$ and $\alpha > 1/2$, introduce a new weight $(w_\alpha(k))_{k\in\mathbb{Z}}$,

$$w_\alpha(k) := \left(1 + |\frac{k}{2}|\right)^\alpha w(k).$$

Notice that $w_\alpha$ is again symmetric and submultiplicative.

LEMMA 2.10. $(\sum_{n\in\mathbb{Z}} w_1(2n)^2\beta_1(n)^2)^{1/2} \leq \|V\|_w^2$.

*Proof.* By Lemmas A.1 and A.2 (in particular, Lemma A.2 for $\alpha = 1$) and (2.26),

$$\left(\sum_{n\in\mathbb{Z}} w_1(2n)^2\beta_1(n)^2\right)^{1/2} \leq \frac{1}{\pi^2}\left\|\frac{\hat{V}}{k} * \frac{\hat{V}}{k}\right\|_{w_1} \leq \frac{6}{\pi^2}\left\|\frac{\hat{V}}{k}\right\|_{w_1}^2 \leq \|\hat{V}\|_w^2. \qquad \square$$

LEMMA 2.11. *For* $|n| \geq n_w := M + \|V\|_w$,

$$(1+|n|)w(2n)\sup_{|z|\leq M}|\beta_2(n,z)| \leq \frac{1}{3}\|V\|_w^2.$$

*Proof.* By (2.25), for any $|z| \leq M$ and $|n| \geq n_w$,

$$\begin{aligned}|\beta_2(n,z)| &= \frac{1}{\pi^2}\left|\left(\frac{\hat{V}}{k} * \frac{\mathcal{S}^{-n}T(Id-T)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}}{k}\right)(2n)\right| \\ &\leq \frac{1}{\pi^2}\sum_{k\neq\pm n}\frac{|\hat{V}(n-k)|}{|n-k|}\frac{|a_{(n)}(n+k)|}{|n+k|},\end{aligned} \tag{2.28}$$

where

$$a_{(n)}(k) := \mathcal{S}^{-n}T(Id-T)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}(k), \quad (k \in \mathbb{Z}\backslash\{-2n, 0\}).$$

Using that $\|T\|_{\mathcal{L}(\ell^2_{\mathcal{S}^n w})} \leq \frac{M+\|V\|_w}{3|n|} = \frac{n_w}{3|n|}$ (Lemma 2.7) and $\|(Id-T)^{-1}\|_{\mathcal{L}(\ell^2_{\mathcal{S}^n_w})} \leq 2$ (Corollary 2.8), we conclude that

$$\|a_{(n)}\|_w \leq \frac{n_w}{3|n|} 2\|V\|_w \leq \frac{2}{3}\|V\|_w. \tag{2.29}$$

As $w_1$ is submultiplicative, we then obtain from (2.28)

$$\begin{aligned}&(1+|n|)w(2n)|\beta_2(n,z)| \\ &\leq \frac{1}{\pi^2}\sum_{k\neq\pm n} 2w(n-k)|\hat{V}(n-k)|2w(n+k)|a_{(n)}(n+k)| \\ &\leq \frac{4}{\pi^2}\|V\|_w\|a_{(n)}\|_w \leq \frac{1}{3}\|V\|_w^2. \qquad \square\end{aligned}$$

Lemma 2.12.

$$\left(\sum_{|n|\geq n_w}(1+|n|)^4 w(2n)^2|\beta_2(n,0)|^2\right)^{1/2} \leq (1+n_w)\|V\|_w^3.$$

*Proof.* By (2.25),

$$\beta_2(n,0) = [\mathcal{S}^{-n}J\hat{V}, D^{-1}BD^{-1}(Id-T_{z=0})^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}.$$

Write $(Id-T_{z=0})^{-1} = Id + T_{z=0}(Id-T_{z=0})^{-1}$ to obtain $\beta_2(n,0) = \beta_3(n)+\beta_4(n)$ with

$$\begin{aligned}\beta_3(n) :=& [\mathcal{S}^{-n}J\hat{V}, D^{-1}BD^{-1}\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}, \\ \beta_4(n) :=& [\mathcal{S}^{-n}J\hat{V}, D^{-1}BD^{-1}\mathcal{S}^n a_{(n)}]_{\mathbb{Z}(n)},\end{aligned}$$

and

$$a \equiv a_{(n)} := \mathcal{S}^{-n}T_{z=0}(Id-T_{z=0})^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}.$$

Let us first treat $\beta_3$. As $w$ is submultiplicative,

$$w(2n) \leq w(n-k)w(n+k) \leq w(n-k)w(n+j)w(k-j),$$

one obtains, for $|n| \geq n_w$,

$$\begin{aligned}&(1+|n|)^2 w(2n)\beta_3(n) \\ &\leq \frac{1}{\pi^4}\sum_{\substack{k\neq\pm n\\ j\neq\pm n}}(1+|n|)^2\frac{w(n-k)|\hat{V}(n-k)|}{|n-k||n+k|}w(k-j)|\hat{V}(k-j)|\frac{w(n+j)|\hat{V}(n+j)|}{|n-j||n+j|} \\ &= \frac{1}{\pi^4}(R_1+R_2+R_3+R_4),\end{aligned} \tag{2.30}$$

where $R_1, R_2, R_3, R_4$ denote the partial sums corresponding to the index sets $I_1, I_2, I_3$, $I_4 \subseteq \{(k,j) \in \mathbb{Z}^2 | k \neq \pm n, j \neq \pm n\}$ defined as follows:

$$\begin{aligned}
I_1 &:= \{|k-n| > |n|; \quad |j-n| > |n| \quad | \quad k, j \neq \pm n\},\\
I_2 &:= \{|k-n| > |n|; \quad |j-n| < |n| \quad | \quad k, j \neq \pm n\},\\
I_3 &:= \{|k-n| < |n|; \quad |j-n| > |n| \quad | \quad k, j \neq \pm n\},\\
I_4 &:= \{|k-n| < |n|; \quad |j-n| < |n| \quad | \quad k, j \neq \pm n\}.
\end{aligned}$$

Let us first estimate $R_2(n)$. For $k, j$ with $|k-n| > |n|, |j-n| < |n|$, one has $1+|n| \leq |k-n|$ and $1+|n| \leq |j+n|$, and thus,

$$\frac{(1+|n|)^2}{|k-n||j+n|} \leq \frac{|k-n|}{|k-n|}\frac{|j+n|}{|j+n|} \leq 1.$$

Therefore, $R_2(n)$ is bounded by

$$\sum_{(k,j)\in I_2} \frac{w(n-k)|\hat{V}(n-k)|}{|n+k|} w(k-j)|\hat{V}(k-j)| \frac{w(n+j)|\hat{V}(n+j)|}{|n-j|}. \tag{2.31}$$

Let $\xi(j) := w(j)|\hat{V}(j)|$. Then

$$\sum_{\substack{|j-n|<|n|\\ j\neq n}} \xi(k-j)\frac{\xi(n+j)}{|n-j|} = \sum_{\substack{|\ell-2n|<|n|\\ 2n-\ell\neq 0}} \xi(k+n-\ell)\frac{\xi(\ell)}{|2n-\ell|} \leq \rho. \tag{2.32}$$

Hence, for $h \in \ell_2(\mathbb{Z})$,

$$\begin{aligned}
&\sum_{|n|\geq n_w} |h(n)|R_2(n) \leq \sum_{n,k,\ell} |h(n)|\frac{\xi(n+k)}{|k+n|}\frac{\xi(k+n-\ell)\xi(\ell)}{|2n-\ell|}\\
&= \sum_{n,k,\ell} \frac{h(n)\xi(k+n-\ell)}{2n-\ell}\frac{\xi(n-k)\xi(\ell)}{|k+n|}\\
&\leq \left(\sum_{n,k,\ell}\frac{|h(n)|^2\xi(k+n-\ell)^2}{|2n-\ell|^2}\right)^{1/2} \left(\sum_{n,k,\ell}\frac{\xi(n-k)^2\xi(\ell)^2}{|k+n|^2}\right)^{1/2}\\
&\leq \left(\sum_{n,j,\ell}\frac{|h(n)|^2\xi(j)^2}{|2n-\ell|^2}\right)^{1/2} \left(\sum_{j,\ell,n}\frac{\xi(j)^2\xi(\ell)^2}{|2n-j|^2}\right)^{1/2}\\
&\leq \cdot 2\|h\|\ \|\xi\| \cdot 2\|\xi\|^2 = 4\|h\|\ \|\xi\|^3.
\end{aligned}$$

Thus we have proved that

$$\left(\sum_{|n|\geq n_w} R_2(n)^2\right)^{1/2} \leq 4\|\xi\|^3 \leq 4\|V\|_w^3.$$

Using the convolution estimate $\|U * V\|_{\ell^2} \leq \|U\|_{\ell^2}\|V\|_{\ell^1}$ one obtains, for $j = 1$, 3, and 4,

$$\left(\sum_{|n|\geq n_w} R_j(n)^2\right)^{1/2} \leq 4\|V\|_w^3.$$

Hence,

$$\left(\sum_{|n|\geq n_w}(1+|n|)^4 w(2n)^2|\beta_3(n)|^2\right)^{1/2}$$
$$\leq \frac{1}{\pi^4}4\cdot 4\cdot \|V\|_w^3 \leq \|V\|_w^3.$$

To estimate $\beta_4(n)$ we proceed similarly. By definition,

$$\beta_4(n)=\sum_{k\neq\pm n}\hat{V}(n\leq k)\sum_{j\neq\pm n}\frac{1}{\pi^2(k^2-n^2)}\hat{V}(k-j)\frac{1}{\pi^2(j^2(j^2-n^2)}a(n+j),$$

whence

$$\begin{aligned}&(1+|n|)^2w(2n)|\beta_4(n)|\\ &\leq \frac{1}{\pi^4}\sum_{\substack{k\neq\pm n\\ j\neq\pm n}}(1+|n|)^2\frac{w(n-k)\hat{V}(n-k)}{|n-k|\;|n+k|}w(k-j)\hat{V}(k-j)\frac{w(n+j)|a(n+j)|}{|n-j|\;|n+j|}\\ &=\frac{1}{\pi^4}(Q_1+Q_2+Q_3+Q_4),\end{aligned}$$

where $Q_1, Q_2, Q_3, Q_4$ denote the partial sums corresponding to the index sets $I_1, I_2, I_3$, $I_4$ defined above. Each of the four terms $Q_i = Q_i(n)(1\leq i\leq 4)$ is estimated in the same way, so we concentrate only on one of them, say, $Q_2$. Similarly as in (2.31) we obtain

$$\sum_{(k,j)\in I_2}\frac{w(n-k)|\hat{V}(n-k)|}{|n+k|}w(k-j)|\hat{V}(k-j)|w(n+j)\frac{|a(n+j)|}{|n-j|}.$$

Let $\eta(j)\equiv\eta_{(n)}(j):=w(j)|a(j)|$ and $\xi(j):=w(j)|\hat{V}(j)|$. Then

$$\sum_{\substack{|j-n|<|n|\\ j\neq n}}\xi(k-j)\frac{\eta(n+j)}{|n-j|}=\sum_{\substack{|\ell-2n|<|n|\\ 2n-\ell\neq 0}}\xi(k+n-\ell)\frac{\eta(\ell)}{|2n-\ell|}\leq\delta_{(n)}(n+k),$$

where

$$\delta_{(n)}(k+n):=\sum_{\ell\neq 2n,0}\xi(k+n-\ell)\frac{\eta_{(n)}(\ell)}{|2n-\ell|}.$$

Using the convolution estimate $\|U*V\|_{\ell^2}\leq\|U\|_{\ell^2}\|V\|_{\ell^1}$ one concludes that (with $2\cdot\sum_{j\geq 1}\frac{1}{j^2}=2\cdot\frac{\pi^2}{6}<4$)

$$\|\delta_{(n)}\|_{\ell^2}\leq\|\eta_{(n)}\|_{\ell^2}\|\xi\|_{\ell^2}\left(\sum_{j\neq 0}\frac{1}{j^2}\right)^{1/2}\leq 2\|\eta_{(n)}\|_{\ell^2}\|\xi\|_{\ell^2}.$$

As $\|T_{Z=0}\|_{\mathcal{L}(\ell^2_{\mathcal{S}^n w})}\leq\frac{n_w}{3|n|}$ (Lemma 2.7) and $\|(Id-I_{Z=0})^{-1}\|_{\mathcal{L}(\ell^2_{\mathcal{S}^n w})}\leq 2$ (Corollary 2.8), one has

$$\|\eta_{(n)}\|_{\ell^2}\leq\frac{nw}{3|n|}1\|V\|_w,\quad(\forall|n|\geq n_w).$$

Hence,

$$\|\delta_{(n)}\|_{\ell^2} \leq \frac{4n_w}{3|n|}\|V\|_w^2, \quad (\forall |n| \geq n_w).$$

This leads to

$$Q_2(n) \leq \sum_{\substack{|k-n|>|n| \\ k\neq -n}} \xi(n-k)\frac{\delta_{(n)}(k+n)}{|k+n|} \leq \|\xi\|_{\ell^2}\frac{4n_w}{3|n|}\|V\|_w^2 \cdot 2,$$

and hence,

$$\begin{aligned}\left(\sum_{|n|\geq w_w} Q_2(n)^2\right)^{1/2} &\leq \frac{8n_w}{3}\|V\|_w^3\left(\sum_{|n|\geq n_w}\frac{1}{n^2}\right)^{1/2} \\ &\leq \frac{16n_w}{3}\|V\|_w^3.\end{aligned}$$

Similar estimates hold for $Q_1, Q_3$, and $Q_4$, and thus,

$$\begin{aligned}&\left(\sum_{|n|\geq n_w}(1+|n|)^4 w(2n)^2|\beta_4(n)|^2\right)^{1/2} \\ &\leq 4\cdot\frac{1}{\pi^4}\cdot\frac{16n_w}{3}\|V\|_w^3.\end{aligned}$$

Combined with the estimate for $\beta_3(n)$, this leads to the claimed statement. □

LEMMA 2.13.

$$\left(\sum_{|n|\geq n_w}(1+|n|)^4 w(2n)^2 \sup_{|z|\leq M}\left|\frac{d}{dz}\beta(n,z)\right|^2\right)^{1/2} \leq 2(1+n_w)^{1/2}\|V\|_w^2.$$

*Proof.* Let $\eta(n,z) := \frac{d}{dz}\beta(n,z)$ and notice that

$$\eta(n,z) = -[\mathcal{S}^{-n}J\hat{V}, (z-A)^{-2}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}.$$

Recall that $(z-A)^{-1} = -D^{-1}(Id-T)^{-1} = -D^{-1} - D^{-1}T(Id-T)^{-1}$ and thus

$$\begin{aligned}(z-A)^{-2} &= \left(-D^{-1} - D^{-1}T(Id-T)^{-1}\right)(z-A)^{-1} \\ &= D^{-2} + D^{-2}T(Id-T)^{-1} - D^{-1}T(Id-T)^{-1})(z-A)^{-1} \\ &= D^{-2} + D^{-2}T(Id-T)^{-1} + D^{-1}T(Id-T)^{-1}D^{-1}(Id-T)^{-1}.\end{aligned}$$

This is used to write $\eta(n,z)$ as a sum,

$$\eta(n,z) = \eta_1(n) + \eta_2(n,z) + \eta_3(n,z),$$

where

$$\begin{aligned}\eta_1(n) &:= -[\mathcal{S}^{-n}J\hat{V}, D^{-2}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}, \\ \eta_2(n,z) &:= -[\mathcal{S}^{-n}J\hat{V}, D^{-2}T(Id-T)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}, \\ \eta_3(n,z) &:= -[\mathcal{S}^{-n}J\hat{V}, D^{-1}T(Id-T)^{-1}D^{-1}(Id-T)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}.\end{aligned}$$

The three terms $\eta_1, \eta_2$, and $\eta_3$ are estimated separately. The coefficient $\eta_1$ is independent of $z$ and

$$\begin{aligned}\eta_1(n) &= \frac{1}{\pi^4}\left(\frac{\hat{V}}{k^2} * \frac{\hat{V}}{k^2}\right)(2n)\\ &= \frac{1}{\pi^4}\sum_{k\neq\pm n}\frac{\hat{V}(n-k)}{(n-k)^2}\frac{\hat{V}(n+k)}{(n+k)^2}.\end{aligned}$$

Hence, by Lemmas A.1 and A.2 (cf. formula (A.2) for $\alpha = 2$),

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}} w_2(2n)^2\eta_1(n)^2\right)^{1/2} &\leq \frac{1}{\pi^4}\left\|\frac{\hat{V}}{k^2} * \frac{\hat{V}}{k^2}\right\|_{w_2}\\ &\leq \frac{6}{\pi^4}\left\|\frac{\hat{V}}{k^2}\right\|_{w_2}^2 \leq \frac{6}{\pi^4}\|\hat{V}\|_w^2. \qquad (2.33)\end{aligned}$$

To estimate $\eta_2(n,z)$, introduce, for $|z| \leq M, |n| \geq n_w$, and $k \in \mathbb{Z}\backslash\{-2n, 0\}$,

$$a(k,z) \equiv a_{(n)}(k,z) := -\mathcal{S}^{-n}T(Id-T)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}(k).$$

Then $\|T\|_{\mathcal{L}(\ell^2_{\mathcal{S}^n_w})} \leq \frac{n_w}{3|n|}$ (Lemma 2.7) and $\|(Id-T)^{-1}\|_{\mathcal{L}(\ell^2_{\mathcal{S}^n_w})} \leq 2$ (Corollary 2.8); hence

$$\|a_{(n)}\|_w \leq \frac{n_w}{3|n|}2\|V\|_w,$$

and, as $w_2(n) = (1+|n|)^2w(2n)$ is submultiplicative,

$$\begin{aligned}&(1+|n|)^2w(2n)|\eta_2(n,z)| \leq \frac{1}{\pi^4}\left|\frac{\hat{V}}{k^2} * \frac{a_{(n)}}{k^2}\right|(2n)\\ &\leq \frac{1}{\pi^4}\sum_{k\neq\pm n}\frac{(1+|n-k|)^2}{|n-k|^2}w(n-k)|\hat{V}(n-k)|\frac{(1+|n+k|)^2}{|n+k|^2}|a_{(n)}(n+k)|\\ &\leq \frac{4^2}{\pi^4}\|V\|_w\|a_{(n)}\|_w \leq \frac{4^2}{\pi^4}\frac{2}{3}\frac{n_w}{|n|}\|V\|_w^2\end{aligned}$$

and

(2.34)

$$\begin{aligned}&\left(\sum_{|n|\geq n_w}(1+|n|)^4w(2n)^2\sup_{|z|\leq M}|\eta_2(n,z)|^2\right)^{1/2}\\ &\leq \frac{4^2}{\pi^4}\frac{2}{3}\|V\|_w^2\left(n_w^2\sum_{|n|\geq n_w}\frac{1}{n^2}\right)^{1/2} \leq \frac{4^2}{\pi^4}\frac{2}{3}(2(1+n_w))^{1/2}\|V\|_w^2 \leq \frac{4^2}{\pi^4}(1+n_w)^{1/2}\|V\|_w^2.\end{aligned}$$

Hence, we used that $N^2\sum_{|n|\geq N}\frac{1}{n^2} \leq 2N^2(\frac{1}{N^2}+\int_N^\infty\frac{dx}{x^2}) = 2(1+N)$. To estimate $\eta_3(n,z)$, we proceed in the same way as for $\eta_2(n,z)$. Introduce, for $|z| \leq M, |n| \geq n_w$, and $k \in \mathbb{Z}\backslash\{-2n, 0\}$,

$$a(k,z) \equiv a_{(n)}(k,z) = \mathcal{S}^{-n}T(Id-T)^{-1}D^{-1}(Id-T)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}(k).$$

Then, by Lemma 2.7 and Corollary 2.8,

$$\|a_{(n)}\|_w \le \frac{n_w}{3|n|} 2 \frac{1}{\pi^2|n|} 2\|V\|_w,$$

where we use that $|D_{kk}^{-1}| = \frac{1}{\pi^2|n^2-k^2|} \le \frac{1}{\pi^2|n|}$ $\forall k \in \mathbb{Z}(n)$. As $w_2$ is submultiplicative,

$$\begin{aligned}
&(1+|n|)^2 w(2n)|\eta_3(n,z)| \le \frac{1+|n|}{\pi^2}\left|\frac{\hat{V}}{k} * \frac{a(n)}{k}\right|(2n)\\
&\le \frac{1+|n|}{\pi^2}\sum_{k\neq\pm n}\frac{1+|n-k|}{|n-k|}w(n-k)|\hat{V}(n-k)|\frac{1+|n-k|}{|n+k|}w(n+k)|a_{(n)}(n+k)|\\
&\le \frac{4}{\pi^2}(1+|n|)\|V\|_w\|a_{(n)}\|_w\\
&\le \frac{4^2}{\pi^4}\frac{1}{3}\frac{|n|+1}{|n|}\frac{n_w}{|n|}\|V\|_w^2
\end{aligned}$$

and, with

$$\begin{aligned}
&\left(\sum_{|n|\ge n_w}\left(\frac{1+|n|}{n^2}\right)^2\right)^{1/2} \le \sqrt{2}\left(\left(\frac{1+n_w}{n_w^2}\right)^2 + \int_{n_w}^{\infty}\left(\frac{1}{x^4}+\frac{2}{x^3}+\frac{1}{x^2}\right)dx\right)^{1/2}\\
&\quad\le 4n_w^{-1/2},
\end{aligned}$$

one obtains

$$\begin{aligned}
&\left(\sum_{|n|\ge n_w}(1+|n|)^4 w(2n)^2 \sup_{|z|\le M}|\eta_3(n,z)|^2\right)^{1/2}\\
&\le \frac{4^3}{\pi^4}\frac{1}{3}\frac{4}{\sqrt{n_w}}n_w\|V\|_w^2 \le \frac{4^4 n_w^{1/2}}{3\pi^4}\|V\|_w^2.
\end{aligned}\tag{2.35}$$

Combining (2.34)–(2.35), we get

$$\begin{aligned}
&\left(\sum_{|n|\ge n_w}(1+|n|)^4 w(2n)^2 \sup_{|z|\le M}\left|\frac{d}{dz}\beta(n,z)\right|^2\right)^{1/2}\\
&\le \frac{6+4^2+4^{4/3}}{\pi^4}(n_w+1)^{1/2}\|V\|_w^2 \le 2(n_w+1)^{1/2}\|V\|_w^2. \qquad \square
\end{aligned}$$

The previous lemmas lead us to the following main result of this section.

Proposition 2.14. *The following statements hold:*

(i) $\left(\sum_{|n|\ge n_w}(1+|n|)^4 w(2n)^2 \sup_{|z|\le M}|\beta_2(n,z)|^2\right)^{1/2} \le 2(1+n_w)^{3/2}\|V\|_w^2$,

(ii) $\left(\sum_{|n|\ge n_w}(1+|n|)^2 w(2n)^2 |\beta_1(n)|^2\right)^{1/2} \le \|V\|_w^2$,

(iii) $\left(\sum_{|n|\ge n_w}(1+|n|)^2 w(2n)^2 \sup_{|z|\le M}|\beta(n,z)|^2\right)^{1/2} \le 2(1+n_w)^{3/2}\|V\|_w^2$.

*Proof.* (i) By (2.23), for $|z|\le M$,

$$|\beta_2(n,z)| \le |\beta_2(n,0)| + M\sup_{|z|\le M}\left|\frac{d}{dz}\beta(n,z)\right|$$

with $M + \|V\|_w = n_w$

$$\begin{aligned}&\left(\sum_{|n|\geq n_w}(1+|n|)^4 w(2n)^2 \sup_{|z|\leq M}|\beta_2(n,z)|^2\right)^{1/2}\\ &\leq \|V\|_w^3 + 2M(1+n_w)^{1/2}\|V\|_w^2\\ &\leq \|V\|_w^2(\|V\|_w + 2M)(1+n_w)^{1/2};\end{aligned}$$

(ii) Lemma 2.10;
(iii) It follows from conditions (i) and (ii). □

**2.7. The $\zeta$-equation.** In this section, we analyze the $\zeta$-equation, stated in (2.18),

$$\zeta^2 - \left(\hat{V}(2n) + \beta(n, z(\zeta))\right)\left(\hat{V}(-2n) + \beta(-n, z(\zeta))\right) = 0. \tag{2.36}$$

Given $M > 0$ and $V \in L^2_0$, define

$$r_n := \max_{\varepsilon=\pm 1}|\hat{V}(\varepsilon 2n)| + \max_{\varepsilon=\pm 1}|\beta_1(\varepsilon n)| + \max_{\substack{|z|\leq M\\ \varepsilon=\pm 1}}|\beta_2(\varepsilon n, z)|, \tag{2.37}$$

and let $n_* := M + \|V\|$. By Lemmas 2.10 and 2.11, applied for the weight $w = 1$,

$$r_n \leq \|V\| + 2\|V\|^2 \quad \forall n \text{ with } |n| \geq n_*.$$

PROPOSITION 2.15. *Assume that $M > 0$ satisfies*

$$3(1 + \|V\|)^2 \leq \frac{M}{4}. \tag{2.38}$$

*Then, for $n \geq n_*, \zeta$-equation* (2.36) *has exactly two (counted with multiplicity) solutions in the disc $\overline{\mathcal{D}_{r_n}}$.*

*Notation.* We label these two solutions by $\zeta_n^+, \zeta_n^-$ in an arbitrary way, but then we keep them fixed.

*Proof.* Clearly, $\zeta^2 = 0$ has precisely two roots in any disc $\mathcal{D}_r$. For $|\zeta| = Kr_n$ with $1 < K < 2$ close to 1 and any $n \geq n_*$, by (2.37) and (2.23),

$$\sup_{|z|\leq M}|\left(\hat{V}(2n) + \beta(n,z)\right)\left(\hat{V}(-2n) + \beta(-n,z)\right)| \leq r_n^2 < |\zeta|^2 \tag{2.39}$$

and $|\zeta| < 2r_n \leq \frac{M}{2}$. Taking into account (2.38), it follows from Proposition 2.6 that $z_n(\zeta) \in \mathcal{D}_M$ depends analytically on $\zeta$ for $|\zeta| < M/2$ and $n \geq n_*$. Therefore, the left side of (2.36), denoted by $g(\zeta)$, is an analytic function of $\zeta$ in $\mathcal{D}_{M/2}$ and, by (2.39),

$$g(\zeta) = \zeta^2 + g_1(\zeta)\ ; \quad |g_1(\zeta)| < |\zeta|^2 \quad \text{for } |\zeta| \leq \frac{M}{2}.$$

Therefore, by Rouché's theorem, (2.36) has precisely two roots in $\mathcal{D}_{Kr_n}$. As the two roots are independent of $K$, and $1 < K < 2$ is arbitrarily close to 1, we conclude that $\zeta_n^\pm \in \overline{\mathcal{D}_{r_n}}$. □

Let, for $n \geq n_*$,

$$z_n^\pm := z(\zeta_n^\pm) = \zeta_n^\pm + \alpha(n, z(\zeta_n^\pm)), \tag{2.40}$$

where $\zeta_n^{\pm}$ are the two solutions of (2.36), given by Proposition 2.15, and define

$$\lambda_n^{\pm} := n^2\pi^2 + z_n^{\pm}. \tag{2.41}$$

Then $\lambda_n^{\pm}$ is a pair of periodic eigenvalues, $\lambda_n^{\pm} \in spec_{per}(-\frac{d^2}{dx^2}+V)$. In the next section we want to deduce estimates for the gap length sequence $(\lambda_n^+ - \lambda_n^-)_{n\geq 1}$.

**2.8. Gap length estimates.**

PROPOSITION 2.16. *Assume that* $M > 0$ *satisfies*

$$3(1+\|V\|_w)^2 \leq \frac{M}{4}.$$

*Then, with* $n_w := M + \|V\|_w$,

$$\left(\sum_{n\geq n_w} w(2n)^2|\lambda_n^+ - \lambda_n^-|^2\right)^{1/2} \leq 8(1+n_w)^{3/2}(1+\|V\|_w)^2.$$

*Remark.* With $N := 13(1+\|V\|_w)^2, K_1 = 500$ and $K_2 = 5$. Proposition 2.16 gives Theorem 1.1.

*Proof.* Notice that, for $n \geq n_w$, by (2.19),

$$|\lambda_n^+ - \lambda_n^-| = |z_n^+ - z_n^-| \leq |\zeta_n^+ - \zeta_n^-| + \sup_{|z|\leq M}\left|\frac{d}{dz}\alpha(n,z)\right| |z_n^+ - z_n^-|. \tag{2.42}$$

By Lemma 2.5, (as $|z| + \|V\| \leq M + \|V\|_w = n_w \leq n$),

$$\left|\frac{d}{dz}\alpha(n,z)\right| \leq \frac{\|V\|^2}{9n^2} \leq \frac{\|V\|^2}{9n_w^2} \leq \frac{1}{9}. \tag{2.43}$$

Substituting the estimate (2.43) into (2.42) yields

$$\frac{1}{2}|z_n^+ - z_n^-| \leq |\zeta_n^+ - \zeta_n^-|.$$

As $|\zeta_n^+ - \zeta_n^-| \leq |\zeta_n^+| + |\zeta_n^-| \leq 2r_n$, with $r_n$ defined by (2.37), we then conclude that, for $n \geq n_w$,

$$|z_n^+ - z_n^-| \leq 4r_n.$$

In view of Proposition 2.14,

$$\left(\sum_{n\geq n_w} w(2n)^2 r_n^2\right)^{1/2} \leq 4\cdot 2(1+n_w)^{3/2}(1+\|V\|_w)^2. \qquad \square$$

**2.9. Gap length asymptotics.** In this section, we obtain the first two terms in the asymptotics of $\lambda_n^+ - \lambda_n^-$ for $n \to \infty$. Let

$$\begin{aligned} &\rho(\pm n) := \hat{V}(\pm 2n) + \beta_1(\pm n), \\ &\eta(z) \equiv \eta(n,z) := \beta_2(-n,z)\rho(n) + \beta_2(n,z)\rho(-n) + \beta_2(-n,z)\beta_2(n,z), \end{aligned} \tag{2.44}$$

where the decomposition $\beta(\pm n, z) = \beta_1(\pm n) + \beta_2(\pm n, z)$ has been defined in (2.24)–(2.25). Then (2.36) can be written as

$$\zeta^2 - \rho(n)\rho(-n) - \eta(z(\zeta)) = 0. \tag{2.45}$$

LEMMA 2.17. *Assume that $M > 0$ satisfies $3(1 + \|V\|_w)^2 \leq \frac{M}{4}$. Then the following estimates hold:*

(i) $|\rho(n)| \leq \frac{M}{4} \quad \forall n$; $\|\varrho\|_w \leq \|V\|_w + \|V\|_w^2$;

(ii) $\left(\sum_{n \geq n_w} (1+|n|)^2 w(2n)^2 \sup_{|z| \leq M} |\eta(n,z)|\right)^{1/2} \leq 4(1+n_w)^{3/2}(1+\|V\|_w)^2$.

*Proof.* (i) By Lemma 2.10, $|\beta_1(n)| \leq \|V\|^2$, and therefore,

$$|\rho(n)| \leq 2(1 + \|V\|)^2 \leq \frac{M}{4}.$$

Moreover, we have $\|\varrho\|_w < \|V\|_w + \|V\|_w^2$. (ii) By the definition of $\eta(n,z)$, and Proposition 2.14(i)

$$\begin{aligned}
&\sum_{n \geq n_w} (1+|n|)^2 w(2n)^2 \sup_{|z| \leq M} |\eta(n,z)| \\
&\leq 2 \left( \sum_{|n| \geq n_w} (1+|n|)^4 w(2n)^2 \sup_{|z|\leq M} |\beta_2(n,z)|^2 \right)^{1/2} 2(1+\|V\|_w)^2 \\
&+ \sum_{n \geq n_w} (1+|n|)^2 w(2n)^2 \sup_{|z| \leq M} |\beta_2(n,z)|^2 \\
&\leq 4(1+\|V\|_w)^2 \cdot 2(1+n_w)^{3/2}(1+\|V\|_w)^2 + 4(1+n_w)^3(1+\|V\|_w)^4 \\
&\leq 12(1+n_w)^3(1+\|V\|_w)^4. \qquad \square
\end{aligned}$$

LEMMA 2.18. *Assume that $M > 0$ satisfies $3(1 + \|V\|_w)^2 \leq \frac{M}{4}$. Then, for $n \geq n_w$, either of the two roots $\hat{\zeta} \in \{\zeta_n^\pm\}$ satisfies*

$$\min_{\pm} |\hat{\zeta} \pm (\rho(n)\rho(-n))^{1/2}| \leq 5 \sup_{|z| \leq M} |\eta(z)|^{1/2}.$$

*Proof.* Choose an arbitrary root $R$ of $R^2 = \rho(n)\rho(-n)$ and let $s := \sup_{|z| \leq M} |\eta(z)|$. We distinguish the following two cases.

*Case* 1: $|R^2| \leq 4s$: we have, with $\hat{z} = z_n^\pm$ for $\hat{\zeta} = \zeta_n^\pm$,

$$\begin{aligned}
|(\hat{\zeta} \pm R)^2| &\leq 2|\hat{\zeta}^2| + 2|R^2| \\
&\leq 2|R^2| + 2|\eta(\hat{z})| + 2|R^2| \\
&\leq 4|R^2| + 2|\eta(\hat{z})| \leq 18s \leq (5s^{1/2})^2.
\end{aligned}$$

*Case* 2: $|R^2| > 4s$: in this case, $|R^2| > 0$ and (2.45) can be rewritten as

$$\zeta^2 = R^2 \left(1 + \frac{\eta(z(\zeta))}{R^2}\right), \tag{2.46}$$

where $z(\zeta)$ is a solution of the $z$-equation (2.17). Let $\xi := \frac{\zeta}{R}$. Then, (2.46) can be written as

$$\xi^2 = 1 + \frac{\eta(z(\zeta))}{R^2}. \tag{2.47}$$

By assumption, $|R^2| > 4s$ and, as $|z(\zeta)| \leq M$, $|\eta(z(\zeta))/R^2| \leq \frac{1}{4}$. Denoting by $(1+w)^{1/2}$ the branch of the square root determined by $1^{1/2} = 1$, we obtain the equations

$$\xi = F_{\pm}(\xi), \tag{2.48}$$

where $F_{\pm}(\xi) := \pm(1+\frac{\eta(z)}{R^2})^{1/2}$, with $z \equiv z(R\xi)$. Let us first consider the equation $\xi = F_+(\xi)$. Let $\mathcal{D}_{\frac{1}{4}}(1) := \{\xi \in \mathbb{C} \quad | \quad |\xi - 1| < \frac{1}{4}\}$ and notice that, for $\xi \in \overline{\mathcal{D}_{\frac{1}{4}}(1)}, \zeta = R\xi$ satisfies $|\zeta| \leq \frac{M}{4}\frac{5}{4} < \frac{M}{2}$, where we used the estimate $|R| \leq \frac{M}{4}$ of Lemma 2.17(i). According to Proposition 2.6, $z = \zeta + \alpha(n,z)$ has a unique solution $z(\zeta) \in \mathcal{D}_M$. This shows that $F_+(\xi) = (1+\frac{\eta(z(R\xi))}{R^2})^{1/2}$ is well defined for $\xi \in \overline{\mathcal{D}_{1/4}(1)}$.

As $|(1+x)^{1/2} - 1| \leq \frac{2}{3}|x|$ for $x \in \overline{\mathcal{D}_{1/4}(0)}$ and $|R^2| > 4s$, we conclude that $F_+$ maps $\overline{\mathcal{D}_{1/4}(1)}$ into itself. Furthermore, $F_+$ is continuous, and therefore, by Brower's fixed point theorem, $\xi = F_+(\xi)$ admits at least one fixed point $\xi^I \in \overline{\mathcal{D}_{1/4}(1)}$,

$$\xi^I = F_+(\xi^I) = \left(1 + \frac{\eta(z^I)}{R^2}\right)^{1/2},$$

where $z^I = z(R\xi^I)$ and $\xi^I$ satisfies the estimate

$$|\xi^I - 1| \leq \left|\left(1+\frac{\eta(z^I)}{R^2}\right)^{1/2} - 1\right| \leq \frac{2}{3}\left|\frac{\eta(z^I)}{R^2}\right| \leq \frac{2}{3}\cdot\frac{1}{2}\frac{s^{1/2}}{|R|},$$

where, for the last inequality, we used that $|R^2| > 4s$. Hence, $\zeta^I := R\xi^I$ satisfies

$$|\zeta^I - R| \leq \frac{1}{2}\sup_{|z|\leq M}|\eta(z)|^{1/2} = \frac{1}{2}s^{1/2}.$$

The same arguments can be used to show that there exists a solution $\xi^{II} \in \overline{\mathcal{D}_{1/4}(-1)}$ of the equation $\xi = F_-(\xi)$ so that $\zeta^{II} := R\xi^{II}$ satisfies

$$|\zeta^{II} + R| \leq \frac{1}{2}\sup_{|z|\leq M}|\eta(z)|^{1/2} = \frac{1}{2}s^{1/2}.$$

Therefore, with $2|R| \geq 4s^{1/2}$

$$\begin{aligned} |\zeta^I - \zeta^{II}| &= |2R - (R-\zeta^I) - (\zeta^{II}+R)| \\ &\geq 2|R| - \frac{1}{2}S^{1/2} - \frac{1}{2}S^12 \geq 3S^{1/2} > 0; \end{aligned} \tag{2.49}$$

hence, $\zeta^I \neq \zeta^{II}$. Moreover, $\zeta^I$ and $\zeta^{II}$ are solutions of (2.45) and thus satisfy, in view of (2.36),

$$|\zeta^I|, |\zeta^{II}| \leq r_n := \max_{\pm}|\hat{V}(\pm 2n)| + \max_{\pm}|\beta_1(\pm n)| + \max_{|z|\leq M}|\beta_2(\pm n, z)|.$$

Therefore, by Proposition 2.15, $\{\zeta^I, \zeta^{II}\} = \{\zeta_n^+, \zeta_n^-\}$. $\square$

For later use, we state the following application of Lemma 2.18.

Corollary 2.19. *Let $V \in H_0^w$ be a 1-periodic potential. Then for $M$ with $3(1+\|V\|_w)^2 \le \frac{M}{4}$,*

$$\left( \sum_{h \ge n_w} w(2n)^2 |\zeta_n^\pm|^2 \right)^{1/2} \le 9(1+n_w)^{3/2}(1+\|V\|_w)^4.$$

*Proof.* By (2.45)

$$\begin{aligned}
&\left( \sum_{n \ge n_w} w(2n)^2 |\zeta_n^\pm|^2 \right)^{1/2} \\
&\le \left( \sum_{n \ge n_w} w(2n)^2 |\rho(n)\rho(-n)| \right)^{1/2} + \left( \sum_{n \ge n_w} w(2n)^2 \sup_{|z| \le M} |\eta(n,z)| \right)^{1/2}.
\end{aligned}$$

By Lemma 2.17,

$$\left( \sum_{n \ge n_w} w(2n)^2 \sup_{|z|\le M} |\eta(n,z)| \right)^{1/2} \le 4(1+n_w)^{3/2}(1+\|V\|_w)^2$$

and, with $\rho(n) = \hat{V}(2n) + \beta_1(n)$,

$$\begin{aligned}
&\left( \sum_{n \ge n_w} w(2n)^2 |\rho(n)\rho(-n)| \right)^{1/2} \\
&\le \left( \sum_{n \ge n_w} w(2n)^2 |\rho(n)|^2 \right)^{1/2} \left( \sum_{n \ge n_w} w(2n)^2 |\rho(-n)|^2 \right)^{1/2} \\
&\le (\|V\|_w + \|V\|_w^2)^2 \le (1+\|V\|_w)^4,
\end{aligned}$$

where we have used Lemma 2.17. ∎

Recall that $\lambda_n^\pm = n^2\pi^2 + z_n^\pm$ denote periodic eigenvalues of the operator $-\frac{d^2}{dx^2} + V$ and $\rho(\pm n)$ have been defined in (2.44).

Theorem 2.20. *Let $V \in H_0^w$ be 1-periodic. Then, for any $M > 0$ with $3(1+\|V\|_w)^2 \le \frac{M}{4}$ and $n_w = M + \|V\|_w$,*

$$\left( \sum_{n \ge n_w} (1+|n|)^2 w(2n)^2 \left( \min_{\pm} \left( (\lambda_n^+ - \lambda_n^-) \pm 2(\rho(n)\rho(-n))^{1/2} \right) \right)^2 \right)^{1/2}$$

$$\le 50(1+n_w)^{3/2}(1+\|V\|_w)^4. \tag{2.50}$$

*Remark.* With $N := 13(1+\|V\|_w)^2, K_3 := 10^6$, and $K_4 := 14$, Theorem 2.20 gives Theorem 1.2.

*Proof.* For $n \ge n_w, \lambda_n^+ - \lambda_n^- = z_n^+ - z_n^-$. Furthermore, $z_n^\pm = \zeta_n^\pm + \alpha(n, z_n^\pm)$ and, by

Lemma 2.18 $\min_\pm |(\zeta_n^+ - \zeta_n^-) \pm 2(\rho(n)\rho(-n))^{1/2}| \le 10 \sup_{|z|\le M} |\eta(n,z)|^{1/2}$. Therefore,

$$
\begin{aligned}
&\min_\pm |(\lambda_n^+ - \lambda_n^-) \pm 2(\rho(n)\rho(-n))^{1/2}| \\
&\le \min_\pm |(\zeta_n^+ - \zeta_n^-) \pm 2(\rho(n)\rho(-n))^{1/2}| + \left( \sup_{|z|\le M} \left| \frac{d}{dz}\alpha(n,z) \right| \right) |z_n^+ - z_n^-| \\
&\le 10 \sup_{|z|\le M} |\eta(n,z)|^{1/2} + \frac{\|V\|^2}{n^2} |z_n^+ - z_n^-|,
\end{aligned}
$$

where for the last inequality we used Lemma 2.5(ii). By Lemma 2.17(ii) (estimate for $\sup_{|z|\le M} |\eta(n,z)|^{1/2}$) and by Proposition 2.16 (estimate for $|z_n^+ - z_n^-| = |\lambda_n^+ - \lambda_n^-|$),

$$
\begin{aligned}
&\left( \sum_{n\ge n_w} (1+|n|)^2 w(2n)^2 \min_\pm |(\lambda_n^+ - \lambda_n^-) \pm 2(\rho(n)\rho(-n))^{1/2}|^2 \right)^{1/2} \\
&\le 10 \left( 4(1+n_w)^{3/2}(1+\|V\|_w)^2 \right) \\
&+ \|V\|^2 \left( \sum_{n\ge n_w} \frac{(1+n)^2}{n^4} w(2n)^2 |z_n^+ - z_n^-|^2 \right)^{1/2} \\
&\le 40(1+n_w)^{3/2}(1+\|V\|_w)^2 + \|V\|_w^2 8(1+n_w)^{3/2}(1+\|V\|_w)^2 \\
&\le 50(1+n_w)^{3/2}(1+\|V\|_w)^4. \qquad \square
\end{aligned}
$$

## 3. Eigenfunctions and Riesz's spaces.

**3.1. Eigenfunctions.** In this section we review the estimates of the Fourier coefficients of an $L_2$-normalized eigenfunction $f$ corresponding to a periodic eigenvalue $\lambda = n^2\pi^2 + z$ of $L = -\frac{d^2}{dx^2} + V$. $f$ is a 2-periodic function in $H^2_{loc}(\mathbb{R};\mathbb{C})$ satisfying

$$(L-\lambda)f = 0;\ \|f\| = 1.$$

Recall that $x^f := \hat{f}(-n), y^f := \hat{f}(n)$, and $F := (\hat{f}(k))_{k\in\mathbb{Z}(n)}$. By Proposition 2.2, $F = x^f F_+ + y^f F_-$ and, by (2.22), $F_\pm := (z - A_n)^{-1}(\mathcal{S}^{\pm n}\hat{V})_{\mathbb{Z}(n)}$. For $n \ge n_w$, $F_+$ and $F_-$ satisfy the estimates (cf. Corollary 2.9)

$$\|F_+\|_{\ell^2_{\mathcal{S}^n w}} \le \frac{2}{\pi^2 n}\|V\|_w, \tag{3.1}$$

$$\|F_-\|_{\ell^2_{\mathcal{S}^{-n} w}} \le \frac{2}{\pi^2 n}\|V\|_w. \tag{3.2}$$

By the normalization of $f$

$$1 = \int_0^2 f(x)\overline{f(x)}dx = 2\left( |x^f|^2 + |y^f|^2 + \sum_{k\ne\pm n} |\hat{f}(k)|^2 \right). \tag{3.3}$$

In particular, one has

$$|x^f|^2 + |y^f|^2 \le \frac{1}{2}. \tag{3.4}$$

Hence, by Cauchy's inequality $|F(k)| \leq (|x^f|^2 + |y^f|^2)^{1/2}(|F_+(k)|^2 + |F_-(k)|^2)^{1/2}$, we obtain in view of (3.1)–(3.2)

$$\|F\|_{\ell^2(\mathbb{Z}(n))} \leq \left(\frac{1}{2} \cdot 2 \cdot \frac{2}{\pi^2 n}\|V\|\right)^{1/2} \leq \frac{1}{4n}\|V\|_w.$$

Thus, for $n$ with $n \geq n_w$,

$$\|F\|_{\ell^2(\mathbb{Z}(n))} \leq \frac{1}{4}.$$

Together with (3.3)–(3.4), this yields

$$\frac{1}{4} \leq |x^f|^2 + |y^f|^2 \leq \frac{1}{2}. \tag{3.5}$$

We summarize our estimates as follows.

LEMMA 3.1. *Let $V \in H_0^w$ be 1-periodic. Then for any $M > 0$ and $n \geq n_w$ with*

$$3(1 + \|V\|_w)^2 \leq \frac{M}{4};\ n \geq n_w := M + \|V\|_w,$$

*an eigenfunction $f$ with $\|f\| = 1$, corresponding to an eigenvalue $\lambda$ with $|\lambda - n^2\pi^2| \leq M$, has the following properties:*

(i) $f(x) = x^f e^{-in\pi x} + y^f e^{in\pi x} + x^f F_+ + y^f F_-$;

(ii) $\frac{1}{4} \leq |x^f|^2 + |y^f|^2 \leq \frac{1}{2}$;

(iii) $\|F_+\|_{\ell^2_{\mathcal{S}^{n_w}}} \leq \frac{\|V\|_w}{4n}$; $\|F_-\|_{\ell^2_{\mathcal{S}^{-n_w}}} \leq \frac{\|V\|_w}{4n}$.

**3.2. Riesz's spaces.** Given a 1-periodic potential $V \in H_0^w$, let $M > 0$ satisfy $3(1 + \|V\|_w)^2 \leq \frac{M}{4}$. For $n \geq n_w := M + \|V\|_w$, there are precisely two (counted with multiplicity) eigenvalues, $\lambda_n^+$ and $\lambda_n^-$, near $n^2\pi^2$ of $L = -\frac{d^2}{dx^2} + V$. Recall that $specL = specL_{Per^+} \cup specL_{Per^-}$. Denote by $P_n^{\pm}$ the Riesz projectors corresponding to the boundary conditions $Per^{\pm}$ (cf. (1.16)) and let

$$E_{2n} := P_n^+(L^2[0,1]); \quad E_{2n-1} := P_n^-(L^2[0,1]), \quad (\forall n \geq 1).$$

If $\lambda_n^+ \neq \lambda_n^-$ or $\lambda_n^+ = \lambda_n^-$ is of geometric multiplicity 2, there exist two linearly independent eigenfunctions, corresponding to the eigenvalues $\lambda_n^+$ and $\lambda_n^-$, and $E_n$ is given by the linear span of these two eigenfunctions. In the case where $\lambda_n^+ = \lambda_n^-$ is of geometric multiplicity 1, $E_n$ denotes the root space of $\lambda_n^+$. Notice that this case might happen if the potential $V$ is complex-valued. As an example we mention $V = \varepsilon e^{2\pi i x}$ ($\varepsilon \neq 0$ arbitrary). The periodic eigenvalues of $-\frac{d^2}{dx^2} + \varepsilon e^{2\pi i x}$ (considered on the interval $[0,2]$) are given by $n^2\pi^2$ ($n \geq 0$), where for every $n \geq 1, n^2\pi^2$ is a double eigenvalue of geometric multiplicity 1 (cf. [3], [5] for details).

Let us describe $E_n$ in the case where $\lambda_n^+ = \lambda_n^-$ is of geometric multiplicity 1 in more detail. Denote by $f$ an $L_2$-normalized eigenfunction corresponding to $\lambda_n^+ = \lambda_n^-$, $Lf = \lambda_n^+ f$. Choose an $L_2$-normalized element $\varphi$ in $E_n$, orthogonal to $f$. Then $E_n = span(f, \varphi)$ and $\varphi$ satisfies

$$(L - \lambda_n^+)\varphi = \xi_n f$$

with $\xi_n \neq 0$. Denote by $c(x,\lambda)$ and $s(x,\lambda)$ the fundamental solution of $-y'' + Vy = \lambda y$ with

$$c(0,\lambda) = 1, \quad c'(0,\lambda) = 0; \quad s(0,\lambda) = 0, \quad s'(0,\lambda) = 1. \tag{3.6}$$

Lemma 3.2. *Assume $\lambda_{2n} = \lambda_{2n-1}$ and $\int_0^1 s(x, \lambda_{2n})^2 dx \neq 0$. Then $\lambda_{2n}$ is of geometric multiplicity 2 iff $\lambda_{2n}$ is a Dirichlet eigenvalue of the operator $L$ on $[0,1]$.*

*Proof.* Assume that $\lambda \equiv \lambda_{2n}$ is of geometric multiplicity 2. Then the fundamental solutions $c(x,\lambda)$ and $s(x,\lambda)$ are eigenfunctions, both either periodic or antiperiodic, and $s(0,\lambda) = 0$. It follows that $s(1,\lambda) = 0$, and therefore, $\lambda$ is a Dirichlet eigenvalue. Conversely, assume that $\lambda \in spec_{Dir}(L)$ is a double periodic eigenvalue. Then $\Delta(\lambda) := c(1,\lambda) + s'(1,\lambda) = \pm 2$ and $\dot{\Delta}(\lambda) := \frac{d}{d\lambda}\Delta(\lambda) = 0$, as well as

$$s(1,\lambda) = 0. \tag{3.7}$$

By the Wronskian identity,

$$1 = c(1,\lambda)s'(1,\lambda) - c'(1,\lambda)s(1,\lambda) = c(1,\lambda)s'(1,\lambda)$$

and, combined with $\Delta(\lambda) = \pm 2$, one obtains

$$c(1,\lambda) = s'(1,\lambda) = \pm 1. \tag{3.8}$$

Take the derivative of the Wronskian identity with respect to $\lambda$ and use $\dot{\Delta}(\lambda) = 0$ to conclude that

$$\begin{aligned} 0 =& \dot{c}(1,\lambda)c'(1,\lambda) + c(1,\lambda)\dot{s}'(1,\lambda) \\ & -\dot{c}'(1,\lambda)s(1,\lambda) - c'(1,\lambda)\dot{s}(1,\lambda) \\ =& \pm\left(\dot{c}(1,\lambda) + \dot{s}'(1,\lambda)\right) - c'(1,\lambda)\dot{s}(1,\lambda) \\ =& \, 0 - c'(1,\lambda)\dot{s}(1,\lambda). \end{aligned}$$

As $\lambda \in spec_{Dir}(L)$, and $\int_0^1 s(x,\lambda)^2 dx \neq 0$, $\dot{s}(1,\lambda) \neq 0$, and therefore,

$$c'(1,\lambda) = 0, \tag{3.9}$$

i.e., $\lambda$ is a Neumann eigenvalue of the operator $L$ on $[0,1]$. By (3.6)–(3.9), $c(x,\lambda)$ and $s(x,\lambda)$ are both periodic eigenfunctions of $L$ on $[0,2]$; hence, $\lambda$ has geometric multiplicity 2. □

**3.3. Orthonormal basis of $E_n$.** In this section we obtain properties for an orthonormal basis $f, \varphi$ of the two-dimensional subspace $E_n$ introduced above ($n \geq n_w$, where $n_w := M + \|V\|_w$). Here $f$ is an eigenfunction of $L = -\frac{d^2}{dx^2} + V$, with $\|f\|_{L^2} = 1$, corresponding to the eigenvalue $\lambda^+ \equiv \lambda_n^+$

$$Lf = \lambda^+ f \tag{3.10}$$

and $\varphi$ is an element in $E_n$ with

$$\langle \phi, f \rangle = 0; \quad \|\varphi\|_{L^2} = 1. \tag{3.11}$$

Notice that $\varphi$ is determined up to a scalar $\kappa \in \{z \in \mathbb{C} \mid |z| = 1\}$. Here $\langle p, q \rangle$ denotes the $L_2$-inner product

$$\langle p, q \rangle = \int_0^2 p(x)\overline{q(x)}dx.$$

In the case when $\lambda^+ \equiv \lambda_n^+$ is a double eigenvalue, $\varphi$ satisfies an equation of the form

$$L\varphi = \lambda^+\varphi + \xi f \quad (\lambda^+ = \text{ double eigenvalue}), \tag{3.12}$$

where $\xi \equiv \xi_n = 0$ if $\lambda_n^+$ has geometric multiplicity 2 and $\xi \neq 0$ if $\lambda^+$ has geometric multiplicity 1. In the case where $\lambda^- \equiv \lambda_n^- \neq \lambda_n^+$, choose a normalized eigenfunction $f^- \equiv f_n^-$ of $\lambda^-$ such that the following holds:

$$0 \leq a := \langle f^-, f\rangle \leq 1; \quad \|f^-\|_{L^2} = 1. \tag{3.13}$$

Write $f^-$ as a linear combination of $f$ and $\varphi$,

$$f^- = af + b\varphi, \tag{3.14}$$

where now the scalar $\kappa$ for $\varphi$ (cf. (3.11)) is chosen in such a way that $0 \leq b$. Then $a^2 + b^2 = 1$, and $b \neq 0$, or

$$a = \cos\theta; \quad b = \sin\theta; \quad 0 < \theta \leq \pi/2. \tag{3.15}$$

The function $\varphi = \frac{1}{b}f^- - \frac{a}{b}f$ satisfies

$$\begin{aligned} L\varphi &= \lambda^- \frac{1}{b}f^- - \lambda^+ \frac{a}{b}f \\ &= \lambda^+ \left(\frac{1}{b}f^- - \frac{a}{b}f\right) + (\lambda^- - \lambda^+)\frac{1}{b}f^- \\ &= \lambda^+\varphi + (\lambda^+ - \lambda^-)\frac{1}{b}(f - f^-) - (\lambda^+ - \lambda^-)\frac{1}{b}f. \end{aligned}$$

Thus, in the case $\lambda^+ \neq \lambda^-$, with $\lambda \equiv \lambda^+$,

$$L\varphi = \lambda\varphi + \xi f + \gamma h, \tag{3.16}$$

where $\gamma \equiv \gamma_n = \lambda_n^+ - \lambda_n^-, h = \frac{1}{b}(f - f^-)$, and $\xi \equiv \xi_n$ is defined by

$$\xi_n := -(\lambda_n^+ - \lambda_n^-)\frac{1}{b} \qquad (\text{case } \lambda_n^+ \neq \lambda_n^-). \tag{3.17}$$

Notice that (3.12) has the same form as (3.16) if we set $h$ equal to 0. It turns out that we will no longer have to treat the following three cases separately.

*Case* 1. $\lambda^+ = \lambda^-$ *and* $\xi = 0$;
*Case* 2. $\lambda^+ = \lambda^-$ *and* $\xi \neq 0$;
*Case* 3. $\lambda^+ \neq \lambda^-$.

The next result shows that the term $\gamma h = (\lambda^+ - \lambda^-)\frac{1}{b}(f - f^-)$ in (3.16) is well under control.

Lemma 3.3. *If $\lambda^+ \neq \lambda^-$, then $b \neq 0$ and $\|h\| \leq \sqrt{2}$.*

*Proof.* By (3.15), $b = sin\theta \neq 0$ for $\lambda^+ \neq \lambda^-$. By (3.14) and (3.15), $h = \frac{1}{b}(f - f^-) = \frac{1-\cos\theta}{\sin\theta}f - \varphi$, and therefore, as $f$ and $\varphi$ are orthogonal,

$$\|h\|^2 = \left|\frac{1-\cos\theta}{\sin\theta}\right|^2 + 1 \leq 2$$

as $\frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}} = \tan\frac{\theta}{2} \leq 1$ for $0 < \theta \leq \frac{\pi}{2}$. ∎

In the remaining part of this section our aim is to obtain estimates for $\xi_n$ (cf. (3.12) and (3.17)). To this end, we write (3.16) in Fourier space. Introduce

$$\varphi = x^\varphi e^{-in\pi x} + y^\varphi e^{in\pi x} + \sum_{k\neq\pm n} \Phi(k)e^{ik\pi x}; \quad \Phi = (\Phi(k))_{k\in\mathbb{Z}(n)},$$

and, similarly

$$h = x^h e^{-in\pi x} + y^h e^{in\pi x} + \sum_{k \neq \pm n} H(k) e^{ik\pi x}, \quad H := (H(k))_{k \in \mathbb{Z}(n)}. \tag{3.18}$$

In view of (2.2)–(2.4), (3.16) leads to the following inhomogeneous system:

$$-zx^\varphi + \hat{V}(-2n) y^\varphi + [\mathcal{S}^n J \hat{V}, \Phi]_{\mathbb{Z}(n)} = \xi_n x^f + \gamma_n x^h, \tag{3.19}$$

$$\hat{V}(2n) x^\varphi - z y^\varphi + [\mathcal{S}^{-n} J \hat{V}, \Phi]_{\mathbb{Z}(n)} = \xi_n y^f + \gamma_n y^h, \tag{3.20}$$

$$(\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} x^\varphi + (\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)} y^\varphi + (A_n - z)\Phi = \xi_n F + \gamma_n H, \tag{3.21}$$

where, as usual, $z \equiv z_n^+ = \lambda_n^+ - n^2\pi^2$.

We use this system to obtain an estimate for $\xi_n$. The sequence $\Phi$ is obtained from (3.21),

$$\begin{aligned} \Phi = {} & (z - A_n)^{-1} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} x^\varphi + (z - A_n)^{-1} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} y^\varphi \\ & - (z - A_n)^{-1} \xi_n F - (z - A_n)^{-1} \gamma_n H \end{aligned} \tag{3.22}$$

and, by (2.11), $F$ is given by

$$F = (z - A_n)^{-1} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} x^f + (z - A_n)^{-1} (\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)} y^f. \tag{3.23}$$

Hence,

$$\begin{aligned} \Phi = {} & (z - A_n)^{-1} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} x^\varphi + (z - A_n)^{-1} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} y^\varphi \\ & - (z - A_n)^{-2} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} \xi_n x^f - (z - A_n)^{-2} (\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)} \xi_n y^f \\ & - (z - A_n)^{-1} \gamma_n H. \end{aligned}$$

In this form, substitute $\Phi$ into (3.19)–(3.20) to obtain, with $\alpha(n,z)$ and $\beta(n,z)$ defined by (2.13) and (2.14),

$$\begin{pmatrix} -z + \alpha(n,z) & \hat{V}(-2n) + \beta(-n,z) \\ \hat{V}(2n) + \beta(n,z) & -z + \alpha(n,z) \end{pmatrix} \begin{pmatrix} x^\varphi \\ y^\varphi \end{pmatrix}$$

$$= \xi_n \begin{pmatrix} x^f - \frac{d}{dz}\alpha(n,z) x^f - \frac{d}{dz}\beta(-n,z) y^f \\ y^f - \frac{d}{dz}\beta(n,z) x^f - \frac{d}{dz}\alpha(n,z) y^f \end{pmatrix}$$

$$+ \gamma_n \begin{pmatrix} x^h + [\mathcal{S}^n J \hat{V}, (z - A_n)^{-1} H] \\ y^h + [\mathcal{S}^{-n} J \hat{V}, (z - A_n)^{-1} H] \end{pmatrix},$$

where we used $\alpha(n,z) = \alpha(-n,z)$, Lemma 2.4, and (cf. Lemma 2.5)

$$\frac{d}{dz}\alpha(n,z) = -[\mathcal{S}^{-n} J \hat{V}, (z - A_n)^{-2} (\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}$$

and

$$\frac{d}{dz}\beta(n,z) = -[\mathcal{S}^{-n} J \hat{V}, (z - A_n)^{-2} (\mathcal{S}^{n} \hat{V})_{\mathbb{Z}(n)}]_{\mathbb{Z}(n)}.$$

Therefore,

$$\xi_n \left( Id_2 - \tfrac{d}{dz} \begin{pmatrix} \alpha(n,z) & \beta(-n,z) \\ \beta(n,z) & \alpha(n,z) \end{pmatrix} \right) \begin{pmatrix} x^f \\ y^f \end{pmatrix} \tag{3.24}$$

$$= \begin{pmatrix} -\zeta_n^+ & \hat{V}(-2n) + \beta(-n,z) \\ \hat{V}(2n) + \beta(n,z) & -\zeta_n^+ \end{pmatrix} \begin{pmatrix} x^\varphi \\ y^\varphi \end{pmatrix}$$

$$-\gamma_n \begin{pmatrix} x^h + [\mathcal{S}^n J\hat{V}, (z - A_n)^{-1}H] \\ y^h + [\mathcal{S}^{-n} J\hat{V}, (z - A_n)^{-1}H] \end{pmatrix},$$

where $\zeta_n^+ = z_n^+ - \alpha(n, z_n^+), Id_2$ denotes the $2 \times 2$ identity matrix, and to simplify notation, $[\cdot,\cdot] = [\cdot,\cdot]_{\mathbb{Z}(n)}$. Let $V \in H_0^w$ be 1-periodic and choose $M > 0$ with $3(1 + \|V\|)^2 \le \frac{M}{4}$. By Lemma 2.5, for any $n \ge n_w := M + \|V\|_w$ and $z = z_n^+$

$$\left| \frac{d}{dz} \alpha(n,z) \right| \le \frac{1}{9} \frac{\|V\|^2}{n^2} \le \frac{1}{9}, \tag{3.25}$$

where we use that, by Lemma 2.5 and Propositions 2.6 and 2.15, $|z_n^+| \le M$. Further, by Lemma 2.13, applied to $w = 1$,

$$(1 + |n|)^2 \left| \frac{d}{dz} \beta(n,z) \right| \le 2(1 + n_w)^{1/2} \|V\|^2, \quad (|n| \ge n_w, \quad |z| \le M).$$

Use that for $|n| \ge n_w$, $\frac{\|V\|^2}{1+|n|} \le \frac{M/12}{n_w} \le \frac{1}{12}$. Thus, for $|n| \ge n_w$ and $z \equiv z_n^+$,

$$\left| \frac{d}{dz} \beta(n,z) \right| \le \frac{1}{6}. \tag{3.26}$$

Combining (3.25) and (3.26), the left-hand side of (3.24) can be estimated from below, for $n \ge n_w$,

$$\begin{aligned} &\left\| \xi_n \left( Id_2 - \frac{d}{dz} \begin{pmatrix} \alpha(n,z) & \beta(-n,z) \\ \beta(n,z) & \alpha(n,z) \end{pmatrix} \right) \begin{pmatrix} x^f \\ y^f \end{pmatrix} \right\|^2 \\ &\qquad \ge |\xi_n|^2 \left( |x^f|^2 \left( 1 - \frac{1}{10} \right) + |y^f|^2 \left( 1 - \frac{1}{10} \right) \right) \ge \left( \frac{1}{3} |\xi_n| \right)^2, \end{aligned} \tag{3.27}$$

where we used that $\frac{1}{4} \le |x^f|^2 + |y^f|^2$ (cf. Lemma 3.1). Further, we need an estimate for $[\mathcal{S}^\pm J\hat{V}, (z - A_n)^{-1}H]$ with $H$ as in (3.18).

LEMMA 3.4. *For $n \ge n_w$,*
(i) $|x^h| \le \sqrt{2}$; $|y^h| \le \sqrt{2}$;
(ii) $|[\mathcal{S}^{\pm n} J\hat{V}, (z - A_n)^{-1}H]| \le \frac{\|V\|}{\pi n}$.

*Proof.* The proof of (i) follows from Lemma 3.3. To prove (ii), notice that for $n \ge n_w$ and $z = z_n^+$,

$$\begin{aligned} &|[\mathcal{S}^{\pm n} J\hat{V}, (z - A_n)^{-1}H]| \\ &\qquad \le \|V\| \|(z - A_n)^{-1}\| \|H\| \le \frac{2}{\pi^2} \frac{1}{n} \|V\| \sqrt{2} \\ &\qquad \le \frac{2}{\pi^2} \frac{1}{n} \|V\| \sqrt{2}, \end{aligned}$$

where we used that $\sqrt{2}\|H\| \le \|h\| \le \sqrt{2}$ (cf. (3.18), Lemma 3.3), and $\|(z-A_n)^{-1}\| \le \frac{2}{\pi^2}\frac{1}{n}$ by Lemma 2.1. ∎

Lemma 3.4 enables us to obtain an estimate *from above* of the right-hand side of (3.24). Lemma 3.4 and estimate (3.27), together with $|x^\varphi|^2 + |y^\varphi|^2 \le 1$, are used to deduce from (3.24) that, for $n \ge n_w = M + \|V\|_w$,

$$\begin{aligned} \frac{1}{3}|\xi_n| \le{} & \Big(|\zeta_n^+| + |\hat{V}(2n)| + |\beta(n,z)|\Big) \\ & + \Big(|\zeta_n^+| + |\hat{V}(-2n)| + |\beta(-n,z)|\Big) + |\gamma_n| \left(\sqrt{2} + \frac{2\|V\|}{\pi n}\right) \\ \le{} & 2|\zeta_n^+| + |\hat{V}(2n)| + |\hat{V}(-2n)| + |\beta(n,z)| + |\beta(-n,z)| + 3|\gamma_n|. \end{aligned} \tag{3.28}$$

Estimate (3.28), combined with earlier estimates for $\beta(\pm n, z)$ and $\gamma_n$, leads to the following inequality:

$$|\xi_n| \le \frac{C}{w(2n)} \qquad \forall n \ge n_w, \tag{3.29}$$

where $C > 0$ depends only on $\|V\|_w$. In fact, the following stronger statement holds.

THEOREM 3.5. *Let $V \in H_0^w$ be a 1-periodic potential and let $M > 0$ satisfy $3(1+\|V\|_w)^2 \le \frac{M}{4}$. Then the sequence $(\xi_n)_{n \ge n_w}$ (cf. (3.12), (3.17)) satisfies*

$$\left(\sum_{n \ge n_w} w(2n)^2 |\xi_n|^2\right)^{1/2} \le 120(1+n_w)^2(1+\|V\|_w)^2$$

*with $n_w := M + \|V\|_w$.*

*Proof.* The terms on the right side of (3.28) are estimated separately. Recall that, by Corollary 2.19,

$$\left(\sum_{n \ge n_w} w(2n)^2 |\zeta_n^+|^2\right)^{1/2} \le 5(1+n_w)^2(1+\|V\|_w)^2, \tag{3.30}$$

by Proposition 2.14,

$$\left(\sum_{|n| \ge n_w} w(2n)^2 \sup_{|z| \le M} |\beta(n,z)|^2\right)^{1/2} \le 2(1+n_w)^2(1+\|V\|_w)^2, \tag{3.31}$$

and by Proposition 2.16, with $|\gamma_n| = |\lambda_n^+ - \lambda_n^-|$,

$$\left(\sum_{n \ge n_w} w(2n)^2 |\gamma_n|^2\right)^{1/2} \le 8(1+n_w)^2(1+\|V\|_w)^2. \tag{3.32}$$

Combining (3.30)–(3.32) with (3.28) leads to the following estimate:

$$\frac{1}{3}\left(\sum_{n \ge n_w} w(2n)^2 |\xi_n|^2\right)^{1/2} \le 40(1+n_w)^2(1+\|V\|_w)^2. \qquad ∎$$

**3.4. Restriction of $L$ on $E_n$.** Here we summarize the results of sections 3.1–3.3 as a statement on the structure of the restriction of $L$ on the Riesz spaces $E_n$.

PROPOSITION 3.6. *Let $V \in H_0^w$ be a 1-periodic potential. Then, for $n$ sufficiently large, the Riesz space $E_n$ has an orthonormal basis $f \equiv f_n$, $\varphi \equiv \varphi_n$ such that*

$$L_{Per^\varepsilon} f = \lambda_{2n} f; \quad L_{Per^\varepsilon} \varphi = \lambda_{2n}\varphi + \xi_n f + \gamma_n h,$$

*where $\varepsilon \in \{+,-\}$ is $+$ for $n$ even and $-$ for $n$ odd and $h \equiv h_n \in E_n$. Moreover, the following inequalities hold:*

$$\|h\| \le 2; \quad |\xi_n| \le \frac{C}{w(2n)}; \quad |\gamma_n| \le \frac{C}{w(2n)}$$

*with $C > 0$ independent of $n$. (For stronger estimates, cf.* (3.30) *and* (3.32)*.)*

## 4. Dirichlet spectrum.

**4.1. Candidates for Dirichlet eigenfunctions.** In section 3.2 we have introduced, for $n$ sufficiently large, the two-dimensional subspaces $E_n$,

$$E_{2n} = \mathrm{Range}(P_n^+); \quad E_{2n-1} = \mathrm{Range}(P_n^-). \tag{4.1}$$

We have chosen an orthonormal basis $(f, \varphi)$ of $E_n$ with $f$ being a normalized eigenfunction for the eigenvalue $\lambda \equiv \lambda_n^+$,

$$Lf = \lambda f,$$

and we showed that $\varphi$ satisfies an equation of the form

$$L\varphi = \lambda\varphi + \xi f + \gamma h, \tag{4.2}$$

where $\gamma \equiv \gamma_n = \lambda_{2n} - \lambda_{2n-1}, h$ satisfies $\|h\| \le 2$ (cf. Lemma 3.3), and estimates for $\xi$ have been established in Theorem 3.5. The following lemma gives an element $G$ in $E_n$, satisfying Dirichlet boundary conditions.

LEMMA 4.1. *Assume that $3(1 + \|V\|_w)^2 \le \frac{M}{4}$ and $n \ge n_w = M + \|V\|_w$. Then there exists an element $G$ in $E_n$ of the form*

$$G = \alpha f + \beta\varphi; \quad 0 \le \alpha \le 1; \quad |\alpha|^2 + |\beta|^2 = 1 \tag{4.3}$$

*so that*

$$G(0) = 0; \quad G(1) = 0.$$

*Proof.* First consider the case where $f(0) = 0$. Then, as $f$ is either periodic or antiperiodic,

$$f(1) = \pm f(0) = 0. \tag{4.4}$$

Thus $G := f$ has the required properties. If $f(0) \ne 0$, notice that $\tilde{G}(x) := -f(0)\varphi(x) + \varphi(0)f(x)$ is a nonzero element in $E_n$, satisfying Dirichlet boundary conditions. Then $G := \kappa\frac{\tilde{G}}{\|\tilde{G}\|} = \alpha f + \beta\varphi$, with $\kappa \in \mathbb{C}, |\kappa| = 1$, chosen to guarantee $0 \le \alpha \le 1$, has the stated properties. ∎

Using (4.2) and (4.3) one obtains

$$\begin{aligned} LG &= \alpha Lf + \beta L\varphi \\ &= \alpha\lambda f + \beta(\lambda\varphi + \xi f + \gamma h) \\ &= \lambda G + \xi\beta f + \gamma\beta h. \end{aligned} \tag{4.5}$$

For $n \geq n_w$, both $\xi \equiv \xi_n$ and $\gamma \equiv \gamma_n$ are small and $G$ almost looks like a Dirichlet eigenfunction.

In the next sections we prove that $\lambda$, respectively, $G$, are good approximations of the Dirichlet eigenvalue $\mu_n$, respectively, Dirichlet eigenfunction $g$.

**4.2. Fourier block decomposition.** Let $L_{Dir}$ be the closed operator $L_{Dir} = -\frac{d^2}{dx^2} + V$ with domain $domL_{Dir} := \{f \in H^2[0,1] \quad | \quad f(0) = 0; \quad f(1) = 0\}$. In this section, let us fix $n$ with $n \geq \max(n_w, 2K_8(M+1))$ (cf. Lemma 1.4 for $K_8$). $P_{Dir} \equiv P_{n,Dir}$ denotes the Riesz projector

$$P_{Dir} := \frac{1}{2\pi i} \int_{|z-n^2\pi^2|=M} (z - L_{Dir})^{-1} dz$$

acting on $L^2([0,1];\mathbb{C})$. Let $Q_{Dir} := Id - P_{Dir}$. Notice that

$$Q_{Dir} f \in domL_{Dir} \qquad \forall f \in domL_{Dir}, \tag{4.6}$$

$$Q_{Dir} L_{Dir} f = L_{Dir} Q_{Dir} f \qquad \forall f \in domL_{Dir}, \tag{4.7}$$

and

$$Q_{Dir} \cdot P_{Dir} = 0; \quad P_{Dir} \cdot Q_{Dir} = 0; \quad P_{Dir}^2 = P_{Dir}; \quad Q_{Dir}^2 = Q_{Dir}. \tag{4.8}$$

According to Lemma 1.5,

$$\|P_{Dir}\| \leq K_{10},$$

$K_{10}$ being an absolute constant, and, therefore

$$\|Q_{Dir}\| \leq K_{10} + 1.$$

Notice that (cf. (1.13) and (1.16))

$$\text{Range}P_{Dir} = \{ag \mid a \in \mathbb{C}\},$$

where $g$ is an $L^2$-normalized eigenfunction for the Dirichlet eigenvalue $\mu \equiv \mu_n$,

$$L_{Dir} g = \mu g; \qquad \|g\| = 1.$$

As $G$ (cf. Lemma 4.1) is in $dom(L_{Dir})$, it admits a decomposition

$$G = P_{Dir} G + Q_{Dir} G = \kappa g + u, \tag{4.9}$$

where $u \in \text{Range}(Q_{Dir}) \subset dom(L_{Dir})$. Therefore,

$$L_{Dir} G = \kappa\mu g + L_{Dir} u \tag{4.10}$$

and

$$P_{Dir}u = 0; \quad Q_{Dir}u = u. \tag{4.11}$$

Hence, (4.7) implies that $Q_{Dir}L_{Dir}u = L_{Dir}u$, and thus,

$$L_{Dir}u \in \mathrm{Range}(Q_{Dir}). \tag{4.12}$$

On the other hand, by (4.5), $LG = \lambda G + R$, where

$$R := \xi\beta f + \gamma\beta h. \tag{4.13}$$

Thus by (4.9),

$$L_{Dir}G = \lambda\kappa g + \lambda u + P_{Dir}R + Q_{Dir}R. \tag{4.14}$$

The left sides of (4.10) and (4.14) being the same, we conclude that

$$\kappa\mu g + L_{Dir}u = \kappa\lambda g + \lambda u + P_{Dir}R + Q_{Dir}R. \tag{4.15}$$

This equation leads to the following lemma.

LEMMA 4.2. *Assume that* $3(1 + \|V\|_w)^2 \leq \frac{M}{4}$ *and* $n \geq \max(n_w, 2K_8(M+1))$. *Then*

$$\kappa(\mu - \lambda)g = P_{Dir}R, \tag{4.16}$$

$$(L_{Dir} - \lambda)u = Q_{Dir}R. \tag{4.17}$$

*Proof.* Apply $P_{Dir}$ to (4.15). In view of (4.7), (4.8), and (4.11)

$$P_{Dir}L_{Dir}u = P_{Dir}L_{Dir}Q_{Dir}u = P_{Dir}Q_{Dir}L_{Dir}u = 0.$$

Further, use that $P_{Dir}g = g$ and $P_{Dir}Q_{Dir}R = 0$ to conclude the identity (4.16). Similarly, by applying $Q_{Dir}$ to (4.15), the second identity (4.17) is obtained. ∎

**4.3. External equation.** In this section we obtain estimates for the difference $\mu - \lambda$ between the $n$th Dirichlet eigenvalue $\mu \equiv \mu_n$ and the eigenvalue $\lambda \equiv \lambda_{2n}$. Recall that $G = \alpha f + \beta\varphi$ with $|\alpha|^2 + |\beta|^2 = 1$ (cf. (4.3)), $U \equiv U_n = Q_{Dir}G = G - \kappa g$ (cf. (4.9)), $L\varphi = \lambda\varphi + \xi f + \gamma h$ (cf. (4.2)), and $R = \beta(\xi f + \gamma h)$ (cf. (4.13)).

LEMMA 4.3. *Assume that* $3(1 + \|V\|_w)^2 \leq \frac{M}{4}$ *and* $n \geq \max(n_w, 2K_8(M+1))$. *Then*

$$\|u_n\| \leq K_{11}\frac{1}{n}(|\xi_n| + 2|\gamma_n|),$$

*where* $K_8 > 0$ *is the absolute constant from Lemma* 1.4 *and* $K_{11} > 0$ *is the absolute constant from Lemma* 1.6.

*Proof.* Apply Lemma 1.6 to (4.17) to get

$$\|u\| \leq \|(\lambda - L_{Dir})^{-1}Q_{Dir}R\| \leq K_{11}\frac{1}{n}\|R\|.$$

By the definition (4.13) and $|\beta| \leq 1$,

$$\|R\| \leq |\xi| + 2|\gamma|,$$

where we used that $\|f\| = 1$ and $\|h\| \leq \sqrt{2}$ (cf. Lemma 3.3). ∎

From the estimate of $\|u\|$ we obtain an estimate of $\kappa \equiv \kappa_n$ in $G = \kappa g + u$ from below.

Lemma 4.4. *Assume that* $3(1 + \|V\|_w)^2 \leq \frac{M}{4}$. *Then there exists* $N_w$ *with* $N_w \geq \max(1 + n_w, 2K_8(M + 1))$ *so that*

$$|\kappa_n| \geq \frac{1}{2} \qquad \forall n \geq N_w.$$

*Proof.* By (4.9), $|\kappa| = \|\kappa g\| = \|G - u\| \geq \|G\| - \|u\| = 1 - \|u\|$. By Lemma 4.3, $\|u\| \leq K_{11} \frac{1}{n}(|\xi| + 2|\gamma|)$, and by Theorem 3.5

$$|\xi_n| \leq 120(1 + n_w)^2(1 + \|V\|_w)^2 \quad \forall n \geq n_w$$

and (cf. Proposition 2.16)

$$|\gamma_n| \leq 8(1 + n_w)^{3/2}(1 + \|V\|_w)^2.$$

Thus, for $n \geq N_w$, with $N_w$ defined by

$$N_w := 300(K_8 + K_{11})(1 + n_w)^2(1 + \|V\|_w)^2; \quad N_w \geq e, \tag{4.18}$$

it follows that $\|u\| \leq 1/2$, and thus $|\kappa| \geq 1/2$. ∎

**4.4. Estimates for the Dirichlet eigenvalues.** From the identity (4.16) we deduce an estimate for $\mu - \lambda$, using the bound for $\kappa$ established in Lemma 4.4.

Theorem 4.5. *Assume* $3(1 + \|V\|_w)^2 \leq \frac{M}{4}$. *Then, for any* $n \geq N_w$, *with* $N_w$ *given by* (4.18),

$$|\mu_n - \lambda_n^+| \leq 2 \cdot K_{10}(|\xi_n| + 2|\gamma_n|),$$

*where* $K_{10}$ *is an absolute constant given by Lemma* 1.5.

*Proof.* By (4.16)

$$|\kappa||\mu - \lambda| \leq \|P_{Dir}\|\|R\| \leq K_{10}\|R\| \tag{4.19}$$

and, by (4.13)

$$\|R\| \leq |\xi| + 2|\gamma|, \tag{4.20}$$

where we used that $\|f\| = 1, \|h\| \leq \sqrt{2}$, and $|\beta| \leq 1$. Combine the estimates (4.19) and (4.20) with the estimate $|\kappa| \geq \frac{1}{2}$ of Lemma 4.4 to obtain the claimed statement. ∎

Combined with the estimates for $\gamma_n$ (Proposition 2.16) and for $\xi_n$ (Theorem 3.5) we obtain the following theorem.

Theorem 4.6. *Assume* $3(1 + \|V\|_w)^2 \leq \frac{M}{4}$. *Then, with* $N_w$ *given by* (4.18),

$$\left( \sum_{n \geq N_w} w(2n)^2 |\mu_n - \lambda_n^+|^2 \right)^{1/2} \leq 300 K_{10} 13^2 (1 + \|V\|_w)^6.$$

*Proof.* By Theorem 3.5,

$$\left( \sum_{n \geq n_w} w(2n)^2 |\xi_n|^2 \right)^{1/2} \leq 120(1 + n_w)^2(1 + \|V\|_w)^2.$$

By Proposition 2.16

$$\left(\sum_{n\geq n_w} w(2n)^2|\lambda_n^+-\lambda_n^-|^2\right)^{1/2} \leq 80(1+n_w)^{3/2}(1+\|V\|_w)^2.$$

Apply this to Theorem 4.5 to obtain the claimed statements. □

## 5. Spectrum for a special class of boundary conditions.

**5.1. Special class of boundary conditions.** An elementary observation (Lemma 4.1) provided us with a nonzero function $G_n$ in the periodic or antiperiodic 2-dimensional Riesz subspace $E_n$ (cf. (4.1)) which satisfied Dirichlet boundary conditions. If the boundary condition has such a feature, the results of section 4 can be extended. This is explained in section 5.2.

We ask the question which boundary conditions *bc*, given by two linearly independent, homogeneous equations, have the property that, for any $n$, the 2-dimensional subspace $E_n$ contains a nonzero function satisfying these *bc*. Any boundary conditions *bc* for the operator $L = -\frac{d^2}{dx^2} + V$ on $[0,1]$, given by two linearly independent homogeneous equations, is a 2-dimensional subspace $\mathcal{E}$ in

$$\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2 = \{(y_0, y_0';\, y_1, y_1')\},$$

where we think of $y_0 = y(0), y_0' = y'(0), y_1 = y(1)$, and $y_1' = y'(1)$ as given by a solution $y(x) \equiv y(x,\lambda)$ of $Ly = \lambda y$. We want $\mathcal{E}$ to have a nontrivial intersection with both 2-dimensional subspaces

$$\mathcal{E}^+ := \{(y_0, y_0';\, y_1, y_1') \in \mathbb{C}^4 \mid y_0 = y_1;\, y_0' = y_1'\}$$

and

$$\mathcal{E}^- := \{(y_0, y_0';\, y_1, y_1') \in \mathbb{C}^4 \mid y_0 = -y_1;\, y_0' = -y_1'\},$$

i.e., with 2-dimensional planes of periodic and antiperiodic boundary conditions. It implies that

$$\dim(\mathcal{E} \cap \mathcal{E}^+) \geq 1; \quad \dim(\mathcal{E} \cap \mathcal{E}^-) \geq 1.$$

But

$$\mathcal{E}^+ \cap \mathcal{E}^- = \{0\},$$

which is obvious from the definition of $\mathcal{E}^+$ and $\mathcal{E}^-$. Therefore,

$$\dim(\mathcal{E} \cap \mathcal{E}^\pm) = 1 \tag{5.1}$$

and

$$\mathcal{E} \cap \mathcal{E}^+ = \{ze^+ | z \in \mathbb{C}\}; \quad e^+ := (a, b; a, b) \neq 0, \tag{5.2}$$

$$\mathcal{E} \cap \mathcal{E}^- = \{ze^- | z \in \mathbb{C}\}; \quad e^- := (c, d; -c, -d) \neq 0. \tag{5.3}$$

We conclude that

$$\mathcal{E} = \{\xi e^+ + \eta e^- | \xi, \eta \in \mathbb{C}\}.$$

It is easy to see that the orthogonal complement of $\mathcal{E}$ in $\mathbb{C}^4$ is given by

$$\mathcal{E}^\perp = \{\xi\ell_1 + \eta\ell_2 | \xi, \eta \in \mathbb{C}\},$$

where $\ell_1 = (b, -a; b, -a)$ and $\ell_2 = (d, -c; -d, c)$. Hence,

$$\mathcal{E} = \{(y_0, y_0'; y_1, y_1') \in \mathbb{C}^2 \times \mathbb{C}^2 | b(y_0 + y_1) - a(y_0' + y_1') = 0; d(y_0 - y_1) - c(y_0' - y_1') = 0\}.$$

In this way, we come, by necessity, to the following two homogeneous linear equations:

$$\begin{aligned} b(y_0 + y_1) - a(y_0' + y_1') &= 0, \\ d(y_0 - y_1) - c(y_0' - y_1') &= 0. \end{aligned} \tag{5.4}$$

They are linearly independent for any pairs $(a, b) \neq 0,\ (c, d) \neq 0$ given by (5.2) and (5.3).

We can assume without loss of generality that

$$|a|^2 + |b|^2 = 1; \quad |c|^2 + |d|^2 = 1. \tag{5.5}$$

**5.2. Spectrum for $L_{bc}$ with $bc$ of class $\mathcal{B}$.** In this section we consider only *regular boundary conditions* (see [16, section 4.8(b)] of the type (5.4). A simple verification along the definition [16, section 4.8(b), (39)] shows that the *boundary conditions* (5.4) *are regular iff*

$$ac \neq 0 \qquad \text{or} \qquad a = c = 0. \tag{5.6}$$

We denote by $\mathcal{B}$ the class of boundary conditions (5.4) which satisfy (5.5) and (5.6).

*Examples.* (i) Dirichlet $bc : (a, b) = (c, d) = (0, 1)$.

(ii) Neumann $bc : (a, b) = (c, d) = (1, 0)$.

(iii) More generally, if $(a, b) = e^{i\theta}(c, d)$, i.e., $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 0$, and $ac \neq 0$, let $\beta := \frac{b}{a}$. Then the boundary conditions $bc$ (5.4) can be rewritten as

$$y'(0) = \beta y(0); \quad y'(1) = \beta y(1),$$

so $bc$ splits and the conditions at the left and right end points of the interval $[0, 1]$ are the same.

Let us analyze $spec(L_{bc})$ for the potential $V = 0$ and boundary conditions $bc$ from the class $\mathcal{B}$. The domain of $L_{bc}$ is defined as

$$domL_{bc} := \{f \in H^2[0, 1] | (f_0, f_0';\ f_1, f_1') \in (5.4)\}.$$

We write, routinely,

$$\begin{gathered} -f'' = \lambda f; \qquad \lambda = \omega^2; \\ f = p\cos\omega x + q\frac{\sin\omega x}{\omega} \end{gathered}$$

and try to find all $\omega$'s such that, with this $f$, the linear system (5.4) has a nonzero solution $(p, q) \in \mathbb{C}^2$. This leads to the characteristic equation

$$(bd + ac\omega^2)\frac{\sin\omega}{\omega} = 0$$

or

$$(bd + ac\lambda)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = 0.$$

If $a = c = 0$ (Dirichlet $bc$, cf. example (i) above), then

$$spec(L_{bc}) = \{\pi^2 k^2 | k \in \mathbb{Z}_{\geq 1}\}$$

and all eigenvalues are simple. If $ac \neq 0$,

$$spec(L_{bc}) = \{\lambda_0\} \cup \{\pi^2 k^2 | k \in \mathbb{Z}_{\geq 1}\},$$

where $\lambda_0 = -\frac{bd}{ac}$. In this case all eigenvalues are simple, except if $\lambda_0 = \pi^2 k^2$ for some $k \in \mathbb{Z}_{\geq 1}$.

Now we can claim that for any $V \in L^2[0,1]$ and $L = -\frac{d^2}{dx^2} + V$, the operator $L_{bc}$ with $bc$ from the class $\mathcal{B}$ has a discrete spectrum $spec(L_{bc})$ which consists, up to a possible additional eigenvalue $\nu_0$, of a sequence $(\nu_n)_{n\geq 1}$ which we enumerate as in (1.5). Further, the operator $L_{bc}$, its resolvent and Riesz projectors have all the properties stated in Lemmas 1.4–1.6 with obvious semantic adjustments.

Property (5.1) gives a substitute for Lemma 4.1. Now we have all the tools to repeat the constructions and the proofs of section 4 for $bc$ in the class $\mathcal{B}$ to get the following theorem.

THEOREM 5.1. *There exist absolute constants $K_{12}, K_{13}$ such that for any 1-periodic potential $V$ in $H_0^w$ and any $bc$ in the class $\mathcal{B}$,*

$$\sum_{n\geq N} w(2n)^2 |\nu_n - \lambda_{2n}|^2 \leq K_{12}(1 + \|V\|_w)^{K_{13}},$$

*where $N = K_{12}(1 + \|V\|_w)^{K_{13}}$.*

**Appendix.** We present two lemmas used in section 1.7 concerning the convolution operation in sequence spaces. For a weight $v = (v(k))_{k\in\mathbb{Z}}$ let

$$C_v^2 := \sup_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left(\frac{v(m)}{v(k)v(m-k)}\right)^2$$

and denote by $\ell_v^2 \equiv \ell_v^2(\mathbb{Z};\mathbb{C})$ the space of sequences $(a(k))_{k\in\mathbb{Z}}$ with

$$\|a\|_v := \left(\sum_k v(k)^2 |a(k)|^2\right)^{1/2} < \infty.$$

LEMMA A.1. *If $C_v < \infty$, then $\ell_v^2(\mathbb{Z};\mathbb{C})$ is a convolution algebra and*

$$\|a * b\|_v \leq C_v \|a\|_v \|b\|_v.$$

*Remark.* Lemma A.1 is a special case of a much more general result due to Nikolski [17].

*Proof of Lemma* A.1. Let $a, b \in \ell_v^2$ be with norm 1 and define $c = (c(k))_{k\in\mathbb{Z}}$ by

$$c(k) := \sum_j a(k-j)b(j).$$

We have to show that $(v(k)c(k))_{k\in\mathbb{Z}}$ is a sequence in $\ell^2$. Let $\alpha(k) := v(k)|a(k)|$ and $\beta(k) := v(k)|b(k)|$ $(k \in \mathbb{Z})$.

For any sequence $\gamma(k))_{k\in\mathbb{Z}}$ in $\ell^2$,

$$\sum_k v(k)|c(k)||\gamma(k)| \leq \sum_k \sum_j |\gamma(k)| \frac{v(k)}{v(k-j)v(j)} \alpha(k-j)\beta(j)$$
$$\leq \left( \sum_{i,j} |\frac{v(i+j)}{v(i)v(j)}|^2 |\gamma(i+j)|^2 \right)^{1/2} \left( \sum_{i,j} (\alpha(i)\beta(j))^2 \right)^{1/2},$$

where for the last inequality we used Cauchy's inequality in $\ell^2(\mathbb{Z}\times\mathbb{Z})$. As $\sum_{i,j} \alpha(i)^2\beta(j)^2 = 1$ by assumption and

$$\sum_{i,j} \left| \frac{v(i+j)}{v(i)v(j)} \gamma(i+j) \right|^2 \leq \sum_m \sum_j \left| \frac{v(m)}{v(m-j)v(j)} \right|^2 |\gamma(m)|^2 \leq C_v^2 \sum_m |\gamma(m)|^2$$

with $(\gamma(m))_{m\in\mathbb{Z}} \in \ell^2$ arbitrary, it follows that $\|c\|_v \leq C_v$. □

For any submultiplicative weight $(w(k))_{k\in\mathbb{Z}}$ and $\alpha \geq 0$, define

$$w_\alpha(k) := \left(1 + \frac{|k|}{2}\right)^\alpha w(k).$$

Notice that $w_\alpha$ is again submultiplicative.

Lemma A.2. *If $\alpha > 1/2$, then $C_{w_\alpha} < \infty$.*

*Proof of Lemma* A.2. As $w$ is submultiplicative,

$$\frac{w_\alpha(k)}{w_\alpha(j)w_\alpha(k-j)} \leq \frac{(1+\frac{1}{2}|k|)^\alpha}{(1+\frac{1}{2}|j|)^\alpha(1+\frac{1}{2}|k-j|)^\alpha},$$

and therefore,

$$C_{w_\alpha} \leq C(\alpha),$$

where $C(\alpha) < \infty$ is a constant satisfying

$$\sum_j \left( \left(1 + \frac{|j|}{2}\right) \left(1 + |\frac{k-j}{2}|\right) \right)^{-2\alpha} \leq C(\alpha)^2 \left(1 + \frac{|k|}{2}\right)^{-2\alpha}. \tag{A.1}$$

By an elementary computation one could show that

$$C(\alpha)^2 \leq 2(1 + 2^{2\alpha}) \frac{2\alpha+1}{2\alpha-1}. \qquad \square \tag{A.2}$$

## REFERENCES


[1] J. Avron and B. Simon, *The asymptotics of the gap in the Mathieu equation*, Ann. Physics, 134 (1981), pp. 76–84.

[2] B.A. Dubrovin, I.M. Krichever, and S.P. Novikov, *Integrable systems* I, in Encyclopedia Math. Sci., Dynamical Systems IV, V.I. Arnold and S.P. Novikov, eds., Springer, Berlin, 1990, pp. 173–280.

[3] M.G. Gasymov, *Spectral analysis of a class of second-order nonselfadjoint differential operators*, Funct. Anal. Appl., 14 (1980), pp. 14–19.

[4] A. Grigis, *Estimations asymptotiques des intervalles d'instabilité pour l'équation de Hill*, Ann. Sci. École Norm. Sup. (4), 20 (1987), pp. 641–672.

[5] V. Guillemin and A. Uribe, *Hardy functions and the inverse spectral method*, Comm. Partial Differential Equations, 8 (1983), pp. 1455–1474.

[6] E. Harrell, *On the effect of the boundary condition on the eigenvalues of ordinary differential equations*, Amer. J. Math., Supplement 1981, John Hopkins University Press, Baltimore, 1981.

[7] H. Hochstadt, *Estimates on the stability intervals for Hill's equation*, Proc. Amer. Math. Soc., 14 (1963), pp. 930–932.

[8] T. Kappeler and B. Mityagin, *Gap estimates of the spectrum of Hill's equation and action variables for KdV*, Trans. Amer. Math. Soc., 351 (1999), pp. 595–617.

[9] M.V. Keldysh, *On the eigenvalues and eigenfunctions of certain classes of nonselfadjoint equations*, Dokl. Akad. Nauk, 77 (1951), pp. 11–14.

[10] M.V. Keldysh, *On the completeness of the eigenfunctions in certain classes of nonselfadjoint operators*, Uspekhi. Mat. Nauk, 27 (1971), pp. 15–41.

[11] B.M. Levitan and I.S. Sargsian, *Introduction to spectral theory*, Transl. Math. Monogr. 39, AMS, Providence, RI, 1975.

[12] W. Magnus and S. Winkler, *Hill's Equation*, Interscience, New York, 1969.

[13] V.A. Marčenko, *Sturm-Liouville Operators and Applications*, Birkhäuser, Boston, 1986.

[14] A.S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, Transl. Math. Monogr. 71, AMS, Providence, 1988.

[15] H.P. McKean and E. Trubowitz, *Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points*, Comm. Pure Appl. Math., 29 (1976), pp. 143–226.

[16] M.A. Neumark, *Lineare Differentialoperatoren*, Akademie-Verlag, Berlin, 1963.

[17] N. Nikolski, *Selected Problems of Weighted Approximation and Spectral Theory*, Proc. Steklov Inst. Math. 120, AMS, Providence, 1976.

[18] J. Pöschel and E. Trubowitz, *Inverse Spectral Theory*, Academic Press, New York, 1987.

[19] J.J. Sansuc and V. Tkachenko, *Spectral properties of non-selfadjoint Hill's operators with smooth potentials*, in Algebraic and Geometric Methods in Mathematical Physics, A. Boutet de Monvel and V. Marčenko, eds., Kluwer, Dordrecht, 1996, pp. 371–385.

[20] E. Trubowitz, *The inverse problem for periodic potentials*, Comm. Pure Appl. Math., 30 (1977), pp. 321–342.